**Mathematics 1120H – Calculus II: Integrals and Series**
Trent University, Winter 2020
Solution to Assignment #1
Half a rectangle, half a rectangle, by a parabola onward!

Please submit your solutions using Blackboard's assignment module. If that fails, please email your solutions to the instructor (`sbilaniuk@trentu.ca`). Scans or photos of handwritten solutions are perfectly acceptable, so long as they are legible and in some common format. (Combined into a single pdf, for preference.)

Consider the rectangle in the Cartesian plane with corners at $(-1, 0)$, $(1, 0)$, $(1, 10)$, and $(-1, 10)$.
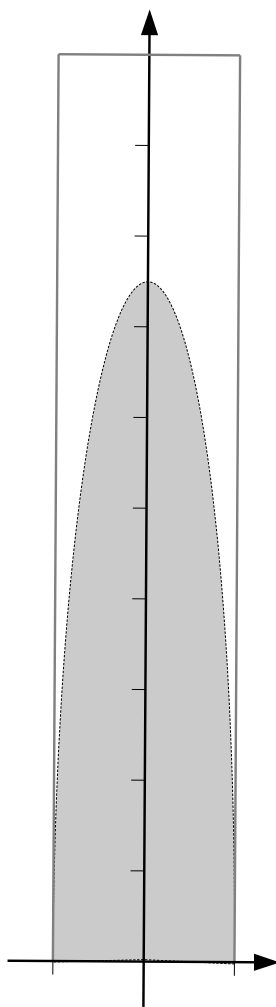
1. Find the equation of the parabola opening downwards that has x-intercepts at -1 and 1 and such that the part of the parabola inside the given rectangle cuts off half the area of the rectangle. *[5]*

SOLUTIONS. A parabola which has x-intercepts at -1 and 1 must have $x - (-1) = x + 1$ and $x - 1$ as factors, so its equation must have the form $y = c(x + 1)(x - 1) = cx^2 - c$ for some constant c. Since this parabola must clearly open downward to cut off anything

less than the whole rectangle, c must be negative. The area the parabola cuts off from the given rectangle is the area between the parabola and the x-axis for $-1 \leq x \leq 1$. This is given by:

$$
\begin{aligned}
\text{Area} = \int_{-1}^{1} \left(cx^2 - c \right) \, dx &= \left(c\frac{x^3}{3} - cx \right)\Big|_{-1}^{1} \\
&= \left(c\frac{1^3}{3} - c1 \right) - \left(c\frac{(-1)^3}{3} - c(-1) \right) \\
&= -\frac{2}{3}c - \frac{2}{3}c = -\frac{4}{3}c
\end{aligned}
$$

The area of the rectangle is $\frac{1}{2}bh = 2 \cdot 10 = 20$, so we need to make the area computed above be half of that, *i.e.* 10. $-\frac{4}{3}c = 10$ implies that $c = -\frac{3}{4} \cdot 10 = -\frac{15}{2} = -7.5$. Thus the desired parabola has the equation $y = -7.5x^2 - (-7.5) = -7.5x^2 + 7.5$. $\square$

2. Find the equation of another parabola, this one opening upwards and passing through $(1, 10)$ and $(-1, 10)$, such that the finite region above this parabola and below the parabola from question bf 1 has one quarter of the area of the rectangle. *[5]*

SOLUTIONS. The basic idea behind the solution to this problem is similar to that of the previous one, complicated by the fact that one must find out where the parabolas intersect to get the limits of the integral one needs to compute, as well as getting a really nasty algebraic equation to solve at the end.

First, it's not hard to see that if you subtracted 10 from the equation of a parabola opening upwards and passing through the points $(-1, 10)$ and $(1, 10)$, it would have to pass through -1 and 1 on the x-axis. It follows that the parabola must have an equation of the form $y = b(x + 1)(x - 1) + 10 = bx^2 - b + 10 = bx^2 + (10 - b)$ for some constant $b > 0$. Our task is to determine b.

Here is a crude sketch of our not-fully-known parabola, $y = bx^2 + (10 - b)$, intersecting with the one we know from question **1**, $y = -7.5x^2 + 7.5$. (Turn the sketch $90° = \pi/2$ radians counterclockwise to get the axes oriented normally. :-)
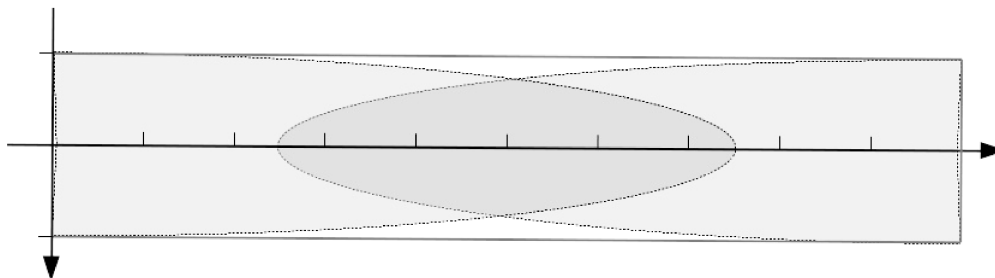
To find the area of the finite region enclosed by the two parabolas we will compute an integral, but to set up the integral we will need to find where the two parabolas intersect. The two parabolas intersect when $bx^2 + (10 - b) = -7.5x^2 + 7.5$. Moving everything to one side in this equation and consolidating gives us the equation $(b + 7.5)x^2 + (2.5 - b) = 0$.

Solving this for x^2 yields $x^2 = -\dfrac{2.5 - b}{b + 7.5} = \dfrac{b - 2.5}{b + 7.5}$, and so $x = \pm\sqrt{\frac{b-2.5}{b+7.5}}$ at the points where the two parabolas intersect. For the sake of brevity, we will write $a = \sqrt{\frac{b-2.5}{b+7.5}} = \left(\frac{b-2.5}{b+7.5}\right)^{1/2}$ in what follows until we really need to expand it.

The area of the finite region between the two parabolas is given by:

$$\text{Area} = \int_{-a}^{a} (\text{upper} - \text{lower})\; dx = \int_{-a}^{a} \left(\left(-7.5x^2 + 7.5\right) - \left(bx^2 + (10 - b)\right)\right)\; dx$$

$$= \int_{-a}^{a} \left(-(7.5 + b)x^2 - (2.5 - b)\right)\; dx = \left.\left(-(7.5 + b)\frac{x^3}{3} - (2.5 - b)x\right)\right|_{-a}^{a}$$

$$= \left(-(7.5 + b)\frac{a^3}{3} - (2.5 - b)a\right) - \left(-(7.5 + b)\frac{(-a)^3}{3} - (2.5 - b)(-a)\right)$$

$$= \left(-(7.5 + b)\frac{a^3}{3} - (2.5 - b)a\right) - \left((7.5 + b)\frac{a^3}{3} + (2.5 - b)a\right)$$

$$= -\frac{2}{3}(7.5 + b)a^3 - 2(2.5 - b)a \quad \ldots \text{ and now we expand } a:$$

$$= -\frac{2}{3}(7.5 + b)\left(\frac{b - 2.5}{b + 7.5}\right)^{3/2} - 2(2.5 - b)\left(\frac{b - 2.5}{b + 7.5}\right)^{1/2}$$

$$= -\frac{2}{3} \cdot \frac{(b - 2.5)^{3/2}}{(7.5 + b)^{1/2}} + 2 \cdot \frac{(b - 2.5)^{3/2}}{(7.5 + b)^{1/2}} = \frac{4}{3} \cdot \frac{(b - 2.5)^{3/2}}{(7.5 + b)^{1/2}}$$

We want this area to be $\frac{1}{4} \cdot 20 = 5$, so we let the expression above equal 5 and solve the equation for b:

$$\frac{4}{3} \cdot \frac{(b - 2.5)^{3/2}}{(7.5 + b)^{1/2}} = 5 \quad \Longrightarrow \quad \frac{16}{9} \cdot \frac{(b - 2.5)^3}{(7.5 + b)} = 25 \quad \Longrightarrow \quad 16(b - 2.5)^3 = 225(7.5 + b)$$

$$\Longrightarrow \quad 16\left(b^3 - 7.5b^2 + 18.75b - 15.625\right) = 1687.5 + 225b$$

$$\Longrightarrow \quad 16b^3 - 120b^2 + 300b - 250 = 225b + 1687.5$$

$$\Longrightarrow \quad 16b^3 - 120b^2 + 75b - 1937.5 = 0$$

Gack! We have to solve a cubic equation with some pretty large coefficients. We *could* look up the cubic formula and continue this by hand, but the smart thing to do would be to get a computer to solve this for us. (Of course, it would have been even smarter to get one to solve the equation

$$\int_{-\left(\frac{2.5-b}{b+7.5}\right)^{1/2}}^{\left(\frac{2.5-b}{b+7.5}\right)^{1/2}} \left(\left(-7.5x^2 + 7.5\right) - \left(bx^2 + (10 - b)\right)\right)\; dx = 5$$

and skip a lot of the work above. Hindsight ... :-) Oh, well let's see what we can do with the cubic equation.

Any decent program that can handle symbolic computation (`Maple`, `Mathematica`, `SageMath`, *etc.*) can do this job. Most of these have graphical user interfaces as a default, but it's easier to reproduce command line work in solutions, so I will use `SageMath` (which is a free and open source program) on the command line. Typing in

```
solve(16*x^3 - 120*x^2 + 75*x + 1937.5 == 0, x)
```

gives the output (line breaks inserted to isolate the three solutions):

```
[x == -5/8*3^(1/3)*(sqrt(141) + 12)^(1/3)*(I*sqrt(3) + 1)
- 5/8*3^(2/3)*(-I*sqrt(3) + 1)/(sqrt(141) + 12)^(1/3) + 5/2,
x == -5/8*3^(1/3)*(sqrt(141) + 12)^(1/3)*(-I*sqrt(3) + 1)
- 5/8*3^(2/3)*(I*sqrt(3) + 1)/(sqrt(141) + 12)^(1/3) + 5/2,
x == 5/4*3^(1/3)*(sqrt(141) + 12)^(1/3)
+ 5/4*3^(2/3)/(sqrt(141) + 12)^(1/3) + 5/2]
```

There are three solutions, two of which are complex numbers: `I` represents the square root of -1, *i.e.* $i = \sqrt{-1}$, in the output. The last solution is a real number, so

$$b = \frac{5}{4} \cdot 3^{1/3} \cdot \left(\sqrt{141} + 12\right)^{1/3} + \frac{5}{4} \cdot 3^{2/3} \cdot \left(\sqrt{141} + 12\right)^{-1/3} .$$

This number is almost certainly irrational in light of the cube roots and $\sqrt{141}$s floating about in the expression. Since it's unlikely to simplify nicely (most computer algebra programs try to simplify such things, not always successfully) we'll get `SageMath` to compute a numerical approximation. Typing

```
N(5/4*3^(1/3)*(sqrt(141) + 12)^(1/3) + 5/4*3^(2/3)/(sqrt(141)
+ 12)^(1/3) + 5/2)
```

into `SageMath` gives us:

```
8.59410268544199
```

Thus $b \approx 8.59410268544199$, and the parabola that does the job is (approximately) given by the equation:

$$y = bx^2 + (10 - b) \approx 8.59410268544199x^2 + 1.40589731455801$$

Enough for a while! ∎