Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Summer 2020
Quiz #3

1. Find the area between $y = x\sin^4(x)$ and the x-axis for $0 \le x \le 2\pi$. Show all your work. *[2]*

SOLUTION. *(Integration by parts and trig reduction formulas.)* First, observe that $x \ge 0$ on the interval $[0, 2\pi]$ and that $\sin^4(x) = (\sin(x))^4 \ge 0$ for all x because it is a fourth power. It follows that $y = x\sin^4(x)$ is never negative on the given interval, so the area between this curve and the x-axis is given by the definite integral $\int_0^{2\pi} x\sin^4(x)\,dx$. How do we go about computing it?

Since the integrand is a product of two dissimilar functions, integration by parts is a natural thing to try. We will give it a go with $u = x$ and $v' = \sin^4(x)$, so $u' = 1$ and $v = \int \sin^4(x)\,dx = ?$. To make this approach work, we first have to work out the trigonometric integral, which we handle with the help of the appropriate reduction formula, which I looked up in the handout *Trigonometric Integrals and Substitutions: A Brief Summary*:

$$\text{If } n \ge 2, \text{ then } \int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx\,.$$

Here goes, we'll be using this reduction formula twice, once with $n = 4$ and once with $n = 2$.

$$\begin{aligned}
v = \int \sin^4(x)\,dx &= -\frac{1}{4}\sin^{4-1}(x)\cos(x) + \frac{4-1}{4}\int \sin^{4-2}(x)\,dx \\
&= -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\int \sin^2(x)\,dx \\
&= -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\left[-\frac{1}{2}\sin^{2-1}(x)\cos(x) + \frac{2-1}{2}\int \sin^{2-2}(x)\,dx\right] \\
&= -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\left[-\frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}\int 1dx\right] \\
&= -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\left[-\frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}x\right] \\
&= -\frac{1}{4}\sin^3(x)\cos(x) - \frac{3}{8}\sin(x)\cos(x) + \frac{3}{8}x
\end{aligned}$$

We left off the usual generic constant of integration since we'll be using this antiderivative as a component in integration by parts. On with the original integral, using the parts above to start:

$$\begin{aligned}
\text{Area} = \int_0^{2\pi} x\sin^4(x)\,dx &= uv\Big|_0^{2\pi} - \int_0^{2\pi} u'v\,dx \\
&= x\left(-\frac{1}{4}\sin^3(x)\cos(x) - \frac{3}{8}\sin(x)\cos(x) + \frac{3}{8}x\right)\Bigg|_0^{2\pi} \\
&\qquad - \int_0^{2\pi} 1\cdot\left(-\frac{1}{4}\sin^3(x)\cos(x) - \frac{3}{8}\sin(x)\cos(x) + \frac{3}{8}x\right)\,dx
\end{aligned}$$

We simplify what we can and break up the remaining integral into digestible pieces:

$$\begin{aligned}\text{Area} &= \left(-\frac{1}{4}x\sin^3(x)\cos(x) - \frac{3}{8}x\sin(x)\cos(x) + \frac{3}{8}x^2\right)\Bigg|_0^{2\pi} \\ &\quad + \frac{1}{4}\int_0^{2\pi} \sin^3(x)\cos(x)\,dx + \frac{3}{8}\int_0^{2\pi} \sin(x)\cos(x)\,dx - \frac{3}{8}\int_0^{2\pi} x\,dx\end{aligned}$$

In the first two remaining integrals we will use the substitution $u = \sin(x)$, so $du = \cos(x)\,dx$, and change the limits as we go along: $\begin{matrix} x & 0 & 2\pi \\ u & 0 & 0 \end{matrix}$. In the last remaining integral, we shall apply the Power Rule. We will also begin evaluating what we have of the antiderivative.

$$\begin{aligned}\text{Area} &= \left(-\frac{1}{4}\cdot 2\pi \cdot \sin^3(2\pi)\cos(2\pi) - \frac{3}{8}\cdot 2\pi \cdot \sin(2\pi)\cos(2\pi) + \frac{3}{8}\cdot(2\pi)^2\right) \\ &\quad - \left(-\frac{1}{4}\cdot 0 \cdot \sin^3(0)\cos(0) - \frac{3}{8}\cdot 0 \cdot \sin(0)\cos(0) + \frac{3}{8}\cdot 0^2\right) \\ &\quad + \frac{1}{4}\int_0^0 u^3\,du + \frac{3}{8}\int_0^0 u\,du - \frac{3}{8}\cdot\frac{x^2}{2}\Bigg|_0^{2\pi} \\ &= \left(-\frac{1}{4}\cdot 2\pi\cdot 0^3 \cdot 1 - \frac{3}{8}\cdot 2\pi \cdot 0 \cdot 1 + \frac{3}{2}\pi^2\right) - (-0 - 0 + 0) \\ &\quad + 0 + 0 - \left(\frac{3}{16}(2\pi)^2 - \frac{3}{16}\cdot 0^2\right) = \frac{3}{2}\pi^2 - \frac{3}{4}\pi^2 = \frac{3}{4}\pi^2\end{aligned}$$

Whew! Note that we used the property of definite integrals that $\int_a^a f(x)\,dx = 0$ no matter what the integrand is to make our lives a bit easier. □

Solution. *(Double angle formulas followed by integration by parts.)* As before, observe that $x \geq 0$ on the interval $[0, 2\pi]$ and that $\sin^4(x) = (\sin(x))^4 \geq 0$ for all x because it is a fourth power. It follows that $y = x\sin^4(x)$ is never negative on the given interval, so the area between this curve and the x-axis is given by the definite integral $\displaystyle\int_0^{2\pi} x\sin^4(x)\,dx$.

We will use the rearranged double angle formulas $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$ or $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$ to simplify powers of sine and cosine.

$$\begin{aligned}\int_0^{2\pi} x\sin^4(x)\,dx &= \int_0^{2\pi} x\left(\sin^2(x)\right)^2\,dx = \int_0^{2\pi} x\left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2\,dx \\ &= \int_0^{2\pi} x\left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(2x)\right)\,dx \\ &= \frac{1}{4}\int_0^{2\pi} x\,dx - \frac{1}{2}\int_0^{2\pi} x\cos(2x)\,dx + \frac{1}{4}\int_0^{2\pi} x\cos^2(2x)\,dx\end{aligned}$$

We will use the Power Rule in the first integral, integration by parts in the second – with $u = x$ and $v' = \cos(2x)$, so $u' = 1$ and $v = \frac{1}{2}\sin(2x)$ – and simplify the third using the other double angle formula.

$$= \frac{1}{4} \cdot \frac{x^2}{2}\Big|_0^{2\pi} - \frac{1}{2}\left[\frac{x}{2}\sin(2x)\Big|_0^{2\pi} - \int_0^{2\pi} 1 \cdot \frac{1}{2}\sin(2x)\,dx\right] + \frac{1}{4}\int_0^{2\pi} x\left(\frac{1}{2} + \frac{1}{2}\cos(4x)\right)\,dx$$

$$= \frac{(2\pi)^2}{8} - \frac{0^2}{8} - \frac{1}{2}\left[\frac{2\pi}{2}\sin(4\pi) - \frac{0}{2}\sin(0) - \frac{1}{2}\left(-\frac{1}{2}\cos(2x)\right)\Big|_0^{2\pi}\right] + \frac{1}{8}\int_0^{2\pi} x\,dx$$
$$+ \frac{1}{8}\int_0^{2\pi} \cos(4x)\,dx$$

$$= \frac{\pi^2}{2} - \frac{1}{2}\left[\pi \cdot 0 - 0 \cdot 0 + \frac{1}{4}\cos(4\pi) - \frac{1}{4}\cos(0)\right] + \frac{1}{8} \cdot \frac{x^2}{2}\Big|_0^{2\pi} + \frac{1}{8} \cdot \frac{1}{4}\sin(4x)\Big|_0^{2\pi}$$

$$= \frac{\pi^2}{2} - \frac{1}{2}\left[\frac{1}{4} \cdot 1 - \frac{1}{4} \cdot 1\right] + \frac{(2\pi)^2}{16} - \frac{0^2}{16} + \frac{1}{32}\sin(8\pi) - \frac{1}{32}\sin(0)$$

$$= \frac{\pi^2}{2} - 0 + \frac{\pi^2}{4} + \frac{1}{32} \cdot 0 - \frac{1}{32} \cdot 0 = \frac{3\pi^2}{4} \qquad \square$$

2. Compute $\displaystyle\int \left(4x^2 + 8x\right)^{5/2}\,dx$. Show all your work. *[3]*

SOLUTION. $\left(4x^2 + 8x\right)^{5/2} = \left(\sqrt{4x^2 + 8x}\right)^5$, so our basic problem is that we're dealing with the square root of a quadratic expression, and one that is not in a form ready-made for a trig substitution. Scanning the previously noted handout *Trigonometric Integrals and Substitutions: A Brief Summary*, we see that it advises one to "complete the square" on the quadratic and even a supplies a formula for doing so:

$$px^2 + qx + r = p\left(x + \frac{q}{2p}\right)^2 + \left(r - \frac{q^2}{4p}\right)$$

Applying this formula to our quadratic yields:

$$4x^2 + 8x = 4x^2 + 8x + 0 = 4\left(x + \frac{8}{2 \cdot 4}\right)^2 + \left(0 - \frac{8^2}{4 \cdot 4}\right)$$
$$= 4(x+1)^2 - 4 = 4\left((x+1)^2 - 1\right)$$

Using this form of the quadratic allows us to simplify the expression inside the square root, first by taking out the factor of 4 and then by substituting for $x+1$:

$$\begin{aligned}
\int \left(4x^2+8x\right)^{5/2}\,dx &= \int \left(\sqrt{4x^2+8x}\right)^5\,dx = \int \left(\sqrt{4\left((x+1)^2-1\right)}\right)^5\,dx \\
&= \int \left(2\sqrt{(x+1)^2-1}\right)^5\,dx = 2^5 \int \left(\sqrt{(x+1)^2-1}\right)^5\,dx \\
&\qquad \text{Substituting } u = x+1, \text{ so } du = dx\text{:} \\
&= 32 \int \left(\sqrt{u^2-1}\right)^5\,du \quad \begin{array}{l}\text{Substituting } u = \sec(t), \\ \text{so } du = \sec(t)\tan(t)\,dt\text{:}\end{array} \\
&= 32 \int \left(\sqrt{\sec^2(t)-1}\right)^5 \sec(t)\tan(t)\,dt \\
&= 32 \int \left(\sqrt{\tan^2(t)}\right)^5 \sec(t)\tan(t)\,dt = 32 \int (\tan(t))^5 \sec(t)\tan(t)\,dt \\
&= 32 \int \tan^6(t)\sec(t)\,dt = 32 \int \left(\tan^2(t)\right)^3 \sec(t)\,dt \\
&= 32 \int \left(\sec^2(t)-1\right)^3 \sec(t)\,dt \\
&= 32 \int \left(\sec^6(t) - 3\sec^4(t) + 3\sec^2(t) - 1\right)\sec(t)\,dt \\
&= 32 \int \left(\sec^7(t) - 3\sec^5(t) + 3\sec^3(t) - \sec(t)\right)\,dt \\
&= 32 \int \sec^7(t)\,dt - 96 \int \sec^5(t)\,dt + 96 \int \sec^3(t)\,dt - 32 \int \sec(t)\,dt
\end{aligned}$$

At this point we look up the reduction formula for secant,

$$\int \sec^n(x)\,dx = \frac{1}{n-1}\tan(x)\sec^{n-2}(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

conveniently among those listed on the previously noted handout, and use it repeatedly, working our way from the higher powers on down.

$$\begin{aligned}
&= 32 \int \sec^7(t)\,dt - 96 \int \sec^5(t)\,dt + 96 \int \sec^3(t)\,dt - 32 \int \sec(t)\,dt \\
&= 32 \left[\frac{1}{6}\tan(t)\sec^5(t) + \frac{5}{6}\int \sec^5(t)\,dt\right] - 96 \int \sec^5(t)\,dt \\
&\qquad + 96 \int \sec^3(t)\,dt - 32 \int \sec(t)\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{32}{6}\tan(t)\sec^5(t) - \frac{416}{6}\int \sec^5(t)\,dt + 96\int \sec^3(t)\,dt - 32\int \sec(t)\,dt \\
&= \frac{16}{3}\tan(t)\sec^5(t) - \frac{208}{3}\left[\frac{1}{4}\tan(t)\sec^3(t) + \frac{3}{4}\int \sec^3(t)\,dt\right] \\
&\qquad + 96\int \sec^3(t)\,dt - 32\int \sec(t)\,dt \\
&= \frac{16}{3}\tan(t)\sec^5(t) - \frac{208}{12}\tan(t)\sec^3(t) + \frac{528}{12}\int \sec^3(t)\,dt - 32\int \sec(t)\,dt \\
&= \frac{16}{3}\tan(t)\sec^5(t) - \frac{52}{3}\tan(t)\sec^3(t) + \frac{132}{3}\left[\frac{1}{2}\tan(t)\sec(t) + \frac{1}{2}\int \sec(t)\,dt\right] \\
&\qquad - 32\int \sec(t)\,dt \\
&= \frac{16}{3}\tan(t)\sec^5(t) - \frac{52}{3}\tan(t)\sec^3(t) + \frac{132}{6}\tan(t)\sec(t) - \frac{60}{6}\int \sec(t)\,dt \\
&= \frac{16}{3}\tan(t)\sec^5(t) - \frac{52}{3}\tan(t)\sec^3(t) + 22\tan(t)\sec(t) - 10\ln\left(\sec(t) + \tan(t)\right) + C
\end{aligned}
$$

Recall that we had substituted $u = \sec(t)$, so $\tan(t) = \sqrt{\sec^2(t) - 1} = \sqrt{u^2 - 1}$, which substitution we now have to undo.

$$
\begin{aligned}
&= \frac{16}{3}\tan(t)\sec^5(t) - \frac{52}{3}\tan(t)\sec^3(t) + 22\tan(t)\sec(t) - 10\ln\left(\sec(t) + \tan(t)\right) + C \\
&= \frac{16}{3}u^5\sqrt{u^2-1} - \frac{52}{3}u^3\sqrt{u^2-1} + 22u\sqrt{u^2-1} - 10\ln\left(u + \sqrt{u^2-1}\right) + C
\end{aligned}
$$

Now recall that we had first substituted $u = x+1$, so $u^2 - 1 = (x+1)^2 - 1 = x^2 + 2x + 1 - 1 = x^2 + 2x$, which substitution we now undo.

$$
\begin{aligned}
&= \frac{16}{3}u^5\sqrt{u^2-1} - \frac{52}{3}u^3\sqrt{u^2-1} + 22u\sqrt{u^2-1} - 10\ln\left(u + \sqrt{u^2-1}\right) + C \\
&= \frac{16}{3}(x+1)^5\sqrt{x^2+2x} - \frac{52}{3}(x+1)^3\sqrt{x^2+2x} + 22(x+1)\sqrt{x^2+2x} \\
&\qquad - 10\ln\left((x+1) + \sqrt{x^2+2x}\right) + C
\end{aligned}
$$

Not a pretty sight, but I'm not motivated to try to improve it . . . :-) ■