TRENT UNIVERSITY, SUMMER 2018

MATH 1110H Test

Monday, 28 May

Time: 50 minutes

Name: Solutions

STUDENT NUMBER: 3141592

Question	Mark
1	___
2	___
3	___
Total	___ /30

Instructions

- *Show all your work.* Legibly, please! Simplify where you reasonably can.
- *If you have a question, ask it!*
- Use the back sides of all the pages for rough work or extra space.
- You may use a calculator and (all sides of) an aid sheet.

1. Compute $\frac{dy}{dx}$ for any *four* (4) of parts **a**–**f**. *[12 = 4 × 3 each]*

a. $y = xe^x$ **b.** $x^2 - y = 1 + x$ **c.** $y = \ln(\cos(x))$

d. $y = \tan\left(x^2\right)$ **e.** $y = \cos(x) + e^{x^2}$ **f.** $y = \frac{x-1}{x^2+1}$

SOLUTIONS. **a.** Product Rule:

$$\frac{dy}{dx} = \frac{d}{dx}\left(xe^x\right) = \left(\frac{d}{dx}x\right) \cdot e^x + x \cdot \left(\frac{d}{dx}e^x\right) = 1 \cdot e^x + x \cdot e^x = (1+x)e^x \quad \Box$$

b. If $x^2 - y = 1 + x$, then $y = x^2 - x - 1$, so $\frac{dy}{dx} = \frac{d}{dx}\left(x^2 - x - 1\right) = 2x + 1 + 0 = 2x + 1$, mostlu using the Power Rule. $\Box$

c. Chain Rule; let $u = \cos(x)$, and then:

$$\frac{dy}{dx} = \frac{d}{dx}\ln(\cos(x)) = \frac{d}{dx}\ln(u) = \left(\frac{d}{du}\ln(u)\right) \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{d}{dx}\cos(x)$$
$$= \frac{1}{cos(x)} \cdot (-\sin(x)) = -\frac{\sin(x)}{\cos(x)} = -\tan(x) \quad \Box$$

d. Chain Rule again, with a bit of Power Rule; let $w = x^2$, and then:

$$\frac{dy}{dx} = \frac{d}{dx}\tan\left(x^2\right) = \frac{d}{dx}\tan(w) = \left(\frac{d}{dw}\tan(w)\right) \cdot \frac{dw}{dx} = \sec^2(w) \cdot \frac{d}{dx}x^2$$
$$= \sec^2\left(x^2\right) \cdot 2x = 2x\sec^2\left(x^2\right) \quad \Box$$

e. Chain Rule with a bit of Power Rule again for the harder part; let $w = x^2$, and then:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos(x) + e^{x^2}\right) = \frac{d}{dx}\cos(x) + \frac{d}{dx}e^{x^2} = -\sin(x) + \frac{d}{dx}e^w$$
$$= -\sin(x) + \left(\frac{d}{dw}e^w\right) \cdot \frac{dw}{dx} = -\sin(x) + e^w \cdot \frac{d}{dx}x^2 = -\sin(x) + e^{x^2} \cdot 2x$$
$$= -\sin(x) + 2xe^{x^2} \quad \Box$$

f. Quotient Rule and a bit of Power Rule:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x-1}{x^2+1}\right) = \frac{\left[\frac{d}{dx}(x-1)\right] \cdot \left(x^2+1\right) - (x-1) \cdot \left[\frac{d}{dx}\left(x^2+1\right)\right]}{\left(x^2+1\right)^2}$$
$$= \frac{[1] \cdot \left(x^2+1\right) - (x-1) \cdot [2x+0]}{\left(x^2+1\right)^2} = \frac{x^2 + 1 - 2x^2 - (-2x)}{\left(x^2+1\right)^2}$$
$$= \frac{-x^2 + 2x + 1}{\left(x^2+1\right)^2} \quad \blacksquare$$

2. Do any *two* (2) of parts **a**–**d**. *[8 = 2 × 4 each]*

a. Compute $\lim_{t \to 0} \dfrac{\tan(t)}{\sin(t)}$.

b. Find the coordinates of the tip of the parabola $y = x^2 - 2x - 3$.

c. Find the equation of the tangent line to $y = x^2 + 1$ at the point $(1, 2)$.

d. Use the ε–δ definition of limits to verify that $\lim_{x \to 1} (4x - 3) = 1$.

SOLUTIONS. **a.** Here goes:

$$\begin{aligned}\lim_{t \to 0} \frac{\tan(t)}{\sin(t)} &= \lim_{t \to 0} \left[\tan(t) \cdot \frac{1}{\sin(t)}\right] = \lim_{t \to 0} \left[\frac{\sin(t)}{\cos(t)} \cdot \frac{1}{\sin(t)}\right] \\ &= \lim_{t \to 0} \frac{1}{\cos(t)} = \frac{1}{\cos(0)} = \frac{1}{1} = 1 \quad \Box\end{aligned}$$

b. *(Completing the square.)* Observe that:

$$\begin{aligned}y = x^2 - 2x - 3 &= x^2 - 2x + \left(\frac{-2}{2}\right)^2 - \left(\frac{-2}{2}\right)^2 - 3 \\ &= \left[x^2 - 2x + (-1)^2\right] + \left[-(-1)^2 - 3\right] = (x-1)^2 - 4\end{aligned}$$

It follows that the tip of the parabola has x-coordinate 1, when $(x - 1)^2$ is as small as possible, and y-coordinate $(1 - 1)^2 - 4 = 0 - 4 = -4$, so the tip is located at $(1, 4)$. $\Box$

b. *(Between the roots.)* The tip of a parabola has x-coordinate halfway between its intercepts, *i.e.* halfway between the roots of the quadratic expression giving the parabola. [Strangely enough, this works even if the roots are complex and so there are no real intercepts!] We can find these roots by either factoring the quadratic, $y = x^2 - 2x - 3 = (x + 1)(x - 3)$, which gives 0 when $x = -1$ or $x = 3$, or by applying the quadratic formula: $x^2 - 2x - 3 = 0$ exactly when

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1} = \frac{2 \pm \sqrt{4 + 12}}{2} = \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2} = 1 \pm 2\,,$$

that is, when $x = 1 - 2 = -1$ or when $x = 1 + 2 = 3$. Either way, the x-coordinate of the tip must be halway between at $x = \dfrac{(-1) + 3}{2} = \dfrac{2}{2} = 1$, and the y-coordinate must then be at $y = 1^2 - 2 \cdot 1 - 3 = 1 - 2 - 3 = -4$, so the tip is at the point $(1, -4)$. $\Box$

b. *(Calculus!)* The tip of a parabola is a maximum or minumum, so the derivative will be 0 at that point. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(x^2 - 2x - 3\right) = 2x - 2 - 0 = 2(x - 1) = 0$ exactly when $x = 1$, so this must be the x-coordinate of the tip. The y-coordinate must then be at $y = 1^2 - 2 \cdot 1 - 3 = 1 - 2 - 3 = -4$, so the tip is at the point $(1, -4)$. $\Box$

c. As a sanity check, $1^2 + 1 = 2$, so the point $(1, 2)$ is indeed on $y = x^2 + 1$. The tangent line to the parabola $y = x^2 + 1$ at x has slope $\frac{dy}{dx} = \frac{d}{dx}\left(x^2 + 1\right) = 2x$; so at the point $(1, 2)$, the slope of the tangent line is $m = 2 \cdot 1 = 2$. It follows that the tangent line has the equation $y = 2x + b$ for some constant b; since it passes through the point $(1, 2)$, $2 = 2 \cdot 1 + b$, so $b = 2 - 2 = 0$. Thus the equation of the tangent line to $y = x^2 + 1$ at the point $(1, 2)$ is $y = 2x$. □

d. According to the ε–δ definition of limits $\lim_{x \to 1} (4x - 3) = 1$ means that for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all x with $|x - 1| < \delta$ we have $|(4x - 3) - 1| < \varepsilon$. To verify this is so, we need to figure out how to find a suitable δ if we are given an $\varepsilon > 0$. We will do so here by reverse-engineering the δ from the desired conclusion:

$$|(4x - 3) - 1| < \varepsilon \iff |4x - 4| < \varepsilon \iff 4\,|x - 1| < \varepsilon \iff |x - 1| < \frac{\varepsilon}{4}$$

Suppose, then that a $\varepsilon > 0$ is given. If we let $\delta = \frac{\varepsilon}{4}$, then any x with $|x - 1| < \delta = \frac{\varepsilon}{4}$ will, by tracing the equivalences above from right to left, have $|(4x - 3) - 1| < \varepsilon$. It follows that $\lim_{x \to 1} (4x - 3) = 1$ by the ε–δ definition of limits. ■

3. Find the domain and any and all intercepts, intervals of increase and decrease, maximum and minimum points, intervals of concavity, and inflection points of the function $g(x) = \dfrac{x+1}{x^2} = \dfrac{1}{x} + \dfrac{1}{x^2}$. *[10]*

SOLUTION. *i. Domain.* $g(x) = \dfrac{x+1}{x^2} = \dfrac{1}{x} + \dfrac{1}{x^2}$ makes sense for all real numbers x except for $x = 0$, so the domain of $g(x)$ is $\{\, x \in \mathbb{R} \mid x \neq 0 \,\} = (-\infty, 0) \cup (0, \infty)$.

ii. Intercepts. $g(0)$ is undefined, so there is no y-intercept. $g(x) = \dfrac{x-1}{x^2} = 0$ only when $x - 1 = 0$, *i.e.* when $x = 1$, so $x = 1$ is the only x-intercept of $g(x)$.

iii. Increase/decrease. First, with a little help from the Quotient and Power Rules:

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(\frac{x+1}{x^2}\right) = \frac{\left[\frac{d}{dx}(x+1)\right] \cdot x^2 - (x+1) \cdot \left[\frac{d}{dx}x^2\right]}{\left(x^2\right)^2} = \frac{1 \cdot x^2 - (x+1) \cdot 2x}{x^4} \\ &= \frac{x^2 - 2x^2 - 2x}{x^4} = \frac{-x^2 - 2x}{x^4} = \frac{-(x+2)}{x^3} \end{aligned}$$

$g'(x) = \dfrac{-(x+2)}{x^3}$ is undefined when $x = 0$, and $g'(x) = 0$ exactly when $x = -2$. When $x < -2$, $x + 2 < 0$ and hence $-(x+2) > 0$, while $x^3 < 0$, so $g'(x) < 0$; when $-2 < x < 0$, $x + 2 > 0$ and hence $-(x+2) < 0$, while $x^3 < 0$, so $g'(x) > 0$; and when $x > 0$, $x + 2 > 0$ and hence $-(x+2) < 0$, while $x^3 > 0$, so $g'(x) < 0$. We summarize this information and the implications for $g(x)$ in the usual table:

x	$(-\infty, -2)$	-2	$(-2, 0)$	0	$(0, \infty)$
$g'(x)$	$-$	0	$+$	undefined	$-$
$g(x)$	$\downarrow$	minimum	$\uparrow$	undefined	$\downarrow$

$g(x)$ is therefore decreasing on $(-\infty, -2)$ and $(0, \infty)$ and increasing on $(-2, 0)$.

iv. Maximum and minimum points. From the table, $g(x)$ has a minimum at $x = -2$; as $g(-2) = \dfrac{-2+1}{(-2)^2} = -\dfrac{1}{4}$, $\left(-2, -\frac{1}{4}\right)$ is the minimum point. Note that $g(x)$ is undefined at $x = 0$, which is the only candidate for a maximum point since it separates an interval of increase from an interval of decrease.

v. Concavity. First, with some more help from the Quotient and Power Rules:

$$\begin{aligned} g''(x) &= \frac{d}{dx} g'(x) = \frac{d}{dx}\left(\frac{-(x+2)}{x^3}\right) = \frac{\left[\frac{d}{dx}\left(-(x+2)\right)\right] \cdot x^3 - \left(-(x+2)\right) \cdot \left[\frac{d}{dx}x^3\right]}{\left(x^3\right)} \\ &= \frac{[-1] \cdot x^3 + (x+2) \cdot \left[3x^2\right]}{x^6} = \frac{-x^3 + 3x^3 + 6x^2}{x^6} = \frac{2x^3 + 6x^2}{x^6} = \frac{2x+6}{x^4} \end{aligned}$$

$g''(x) = \dfrac{2x+6}{x^4} = \dfrac{2(x+3)}{x^4}$ is undefined when $x = 0$, and $g''(x) = 0$ exactly when $x = -3$. When $x < -3$, $2(x+3) < 0$, while $x^4 > 0$, so $g''(x) < 0$; when $-3 < x < 0$, $2(x+3) > 0$, while $x^4 > 0$, so $g''(x) > 0$; and when $x > 0$, $2(x+3) > 0$, while $x^4 > 0$, so $g''(x) > 0$. We summarize this information and the implications for $g(x)$ in the usual table:

x	$(-\infty,-3)$	-3	$(-3,0)$	0	$(0,\infty)$
$g''(x)$	$-$	0	$+$	undefined	$+$
$g(x)$	$\frown$	inflection	$\smile$	undefined	$\smile$

$g(x)$ is therefore concave down on $(-\infty,-3)$ and concave up on $(-3,0)$ and $(0,\infty)$.

vi. Inflection points. From the table, $g(x)$ is defined at $x=-3$ and changes concavity from down to up, so it is an inflection point. Since $g(-3)=\dfrac{-3+1}{(-3)^2}=-\dfrac{2}{9}$, the actual point in question has coordinates $\left(-3,-\frac{2}{9}\right)$. Note that $x=0$ is not an inflection point for two reasons: $g(0)$ is not defined and $g(x)$ is concave up on both sides of $x=0$, so it doesn't change concavity.

vii. Asymptotes. [Not asked for in the question, but it helps when drawing the graph.] First, we check for horizontal asymptotes. Note that as x heads off to ∞ or $-\infty$, $\frac{1}{x}$ and $\frac{1}{x^2}$ both get arbitrarily close to 0.

$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \frac{x+1}{x^2} = \lim_{x\to-\infty}\left(\frac{1}{x}+\frac{1}{x^2}\right) = 0+0=0$$
$$\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} \frac{x+1}{x^2} = \lim_{x\to+\infty}\left(\frac{1}{x}+\frac{1}{x^2}\right) = 0+0=0$$

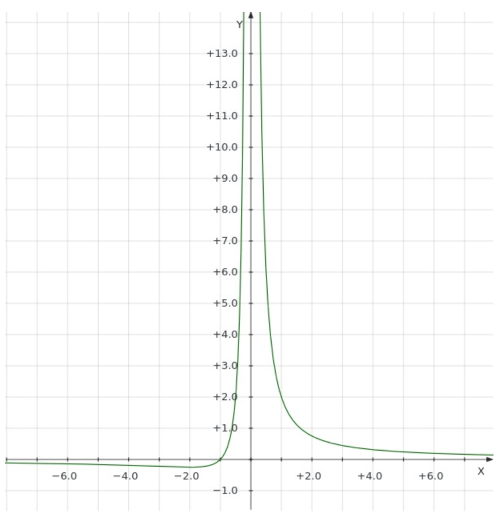
Thus $g(x)$ has the line $y=0$, otherwise known as the x-axis, as a horizontal asymptote in both directions.

Second, we check for vertical asymptotes. Since $g(x)$ is defined and continuous everywhere except at $x=0$, this is the only place vertical asymptotes might occur. Note that as x approaches 0, $x+1$ approaches 1 and x^2 approaches 0 from the positive side.

$$\lim_{x\to0^-} g(x) = \lim_{x\to0^-}\frac{x+1}{x^2} = +\infty \qquad \lim_{x\to0^+} g(x) = \lim_{x\to0^+}\frac{x+1}{x^2} = +\infty$$

Thus $g(x)$ has a vertical asymptote going up on both sides of $x=0$.

viii. The graph. Cheating slightly, by getting a computer to draw it:



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