Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2012

Solution to Assignment #7
Colouring soccer balls*

Mathematical induction is a proof technique that is frequently used in many parts of mathematics, though it doesn't occur all that often in calculus courses. Look it up if you need to and consider the following:

PROPOSITION. *Suppose that at least one soccer ball in a collection of $n \geq 1$ soccer balls is purple. Then all the soccer balls in the collection are purple.*

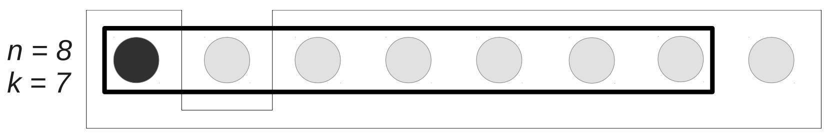
PROOF. We will use the principle of mathematical induction, with n being the number of soccer balls in a given collection:

Base Step: ($n = 1$) If a collection of soccer balls has only one ball, and at least one is purple, then all the balls in the collection are purple. [Yes, it's really that trivial ...]

Induction Hypothesis: ($n = k$ for some $k \geq 1$) If a collection of soccer balls has $n = k$ balls and at least one of them is purple, then all the soccer balls in the collection are purple.

Induction Step: ($n = k + 1$ for some $k \geq 1$) Assume the Induction Hypothesis is true and suppose that a given collection of soccer balls has $n = k + 1$ balls, at least one of which is purple.

Pick a purple ball and call it P. Form a smaller collection of soccer balls by putting aside one of the balls other than P. This smaller collection has $n = k + 1 - 1 = k$ balls, so it follows from the Induction Hypothesis that every ball in the smaller collection is purple. Form a second smaller collection by taking out any one ball in the smaller collection we already have and replacing it with the one set aside earlier. This new smaller collection also has $n = k+1-1 = k$ balls, so it follows from the Induction Hypothesis that every ball in this smaller collection is also purple. Between them, the two smaller collections include every ball in the given collection, so every ball in the given collection is purple.



By the principle of mathematical induction, it follows that if at least one soccer ball in a collection of $n \geq 1$ soccer balls is purple, then all the soccer balls in the collection are purple. ■

1. The conclusion of the proposition given above is clearly absurd. Where exactly does the given proof go wrong, and how? *[10]*

SOLUTION. The (only!) point at which the "proof" breaks down is at the very first application of the inductive step, when $k = 1$ and $n = 2$. Given two balls, one of which is known to be purple, there is no way to group the ball of unknown colour with the purple ball in a subcollection of only one ball ... ■

* Just because the Euro 2012 tournament is on ... :-)