Trent University

MATH 1100Y Test 2

6 July, 2011

Time: 50 minutes

Name: Steffie Graph

Student Number: 01234567

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1	___
2	___
3	___
Total	___

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the extra page and the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Compute any *four* (4) of the integrals in parts **a-f**. *[16 = 4 × 4 each]*

a. $\displaystyle\int \tan^2(x)\,dx$ **b.** $\displaystyle\int_0^{3/2} 2(2x+1)^{3/2}\,dx$ **c.** $\displaystyle\int xe^x\,dx$

d. $\displaystyle\int_0^{\pi} x\cos(x)\,dx$ **e.** $\displaystyle\int \sec^3(x)\tan(x)\,dx$ **f.** $\displaystyle\int_0^1 \left(x^2+2x+3\right)\,dx$

SOLUTION TO **a**. We'll rewrite it using the trig identity $\tan^2(x) = \sec^2(x) - 1$:

$$\int \tan^2(x)\,dx = \int \left(\sec^2(x)-1\right)\,dx = \int \sec^2(x)\,dx - \int 1\,dx = \tan(x) - x + C \quad \square$$

SOLUTION TO **b**. We'll use the substitution $u = 2x+1$, so $du = 2\,dx$ and $\begin{matrix} x & 0 & 3/2 \\ u & 1 & 4 \end{matrix}$.

$$\begin{aligned}\int_0^{3/2} 2(2x+1)^{3/2}\,dx &= \int_1^4 u^{3/2}\,du = \left.\frac{u^{5/2}}{5/2}\right|_1^4 = \frac{1}{5/2}\left(4^{5/2}-1^{5/2}\right) \\ &= \frac{2}{5}\left(2^5-1^5\right) = \frac{2}{5}(32-1) = \frac{2}{5}31 = \frac{62}{5} \quad \square\end{aligned}$$

SOLUTION TO **c**. We'll use integration by parts, with $u = x$ and $v' = e^x$, so $u' = 1$ and $v = e^x$.

$$\int xe^x\,dx = \int uv'\,dx = uv - \int u'v\,dx = xe^x - \int 1e^x\,dx = xe^x - e^x + C \quad \square$$

SOLUTION TO **d**. We will also use integration by parts here, with $u = x$ and $v' = \cos(x)$, so $u' = 1$ and $v = \sin(x)$.

$$\begin{aligned}\int_0^{\pi} x\cos(x)\,dx &= \int_0^{\pi} uv'\,dx = \left.uv\right|_0^{\pi} - \int_0^{\pi} u'v\,dx = \left.x\sin(x)\right|_0^{\pi} - \int_0^{\pi} 1\sin(x)\,dx \\ &= (\pi\sin(\pi) - 0\sin(0)) - \left.(-\cos(x))\right|_0^{\pi} = \pi 0 - 0 + \left.\cos(x)\right|_0^{\pi} \\ &= \cos(\pi) - \cos(0) = -1 - 1 = -2 \quad \square\end{aligned}$$

SOLUTION TO **e**. We'll use the substitution $u = \sec(x)$, so $du = \sec(x)\tan(x)\,dx$.

$$\int \sec^3(x)\tan(x)\,dx = \int \sec^2(x)\sec(x)\tan(x)\,dx = \int u^2\,du = \frac{u^3}{3} + C = \frac{1}{3}\sec^3(x) + C \quad \square$$

SOLUTION TO **f**. The Power Rule is the main tool:

$$\begin{aligned}\int_0^1 \left(x^2+2x+3\right)\,dx &= \int_0^1 x^2\,dx + \int_0^1 2x\,dx + \int_0^1 3\,dx = \left.\frac{x^3}{3}\right|_0^1 + \left.x^2\right|_0^1 + \left.3x\right|_0^1 \\ &= \left(\frac{1^3}{3} - \frac{0^3}{3}\right) + \left(1^2-0^2\right) + (3\cdot 1 - 3\cdot 0) = \frac{1}{3} + 1 + 3 = \frac{13}{3} \quad \square\end{aligned}$$

2. Do any *two* (2) of parts **a-e**. *[12 = 2 × 6 each]*

a. Compute $\int_0^3 \sqrt{9-x^2}\,dx$. What does this integral represent?

b. Sketch the solid obtained by rotating the region bounded by $y=x$, $y=0$, and $x=2$ about the y-axis, and find its volume.

c. Give an example of a function $f(x)$ with $f'(x) = 1 - \int_0^x f(t)\,dt$ for all x.

d. Sketch the region between $y=\sin(x)$ and $y=-\sin(x)$ for $0 \le x \le 2\pi$, and find its area.

e. Compute $\int_1^2 x\,dx$ using the Right-hand Rule.

SOLUTION TO **a**. We will use the substitution $x = 3\sin(\theta)$, so $dx = 3\cos(\theta)\,d\theta$ and $\begin{matrix} x & 0 & 3 \\ \theta & 0 & \pi/2 \end{matrix}$.

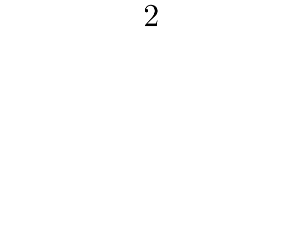
$$\begin{aligned}
\int_0^3 \sqrt{9-x^2}\,dx &= \int_0^{\pi/2} \sqrt{9-9\sin^2(\theta)}\,3\cos(\theta)\,d\theta = \int_0^{\pi/2} 3\sqrt{1-\sin^2(\theta)}\,3\cos(\theta)\,d\theta \\
&= \int_0^{\pi/2} 3\sqrt{\cos^2(\theta)}\,3\cos(\theta)\,d\theta = \int_0^{\pi/2} 3\cos(\theta)\,3\cos(\theta)\,d\theta \\
&= \int_0^{\pi/2} 9\cos^2(\theta)\,d\theta = 9\int_0^{\pi/2} \left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)\,d\theta
\end{aligned}$$

Substitute $u = 2\theta$, so $du = 2\,d\theta$ and $\frac{1}{2}\,du = d\theta$, and $\begin{matrix} \theta & 0 & \pi/2 \\ u & 0 & \pi \end{matrix}$.

$$\begin{aligned}
&= 9\int_0^{\pi} \left(\frac{1}{2}+\frac{1}{2}\cos(u)\right)\frac{1}{2}\,du = \frac{9}{4}\int_0^{\pi} (1+\cos(u))\,du \\
&= \frac{9}{4}(u+\sin(u))\Big|_0^{\pi} = \frac{9}{4}(\pi+\sin(\pi)) - \frac{9}{4}(0+\sin(0)) \\
&= \frac{9}{4}(\pi+0) - \frac{9}{4}(0+0) = \frac{9}{4}\pi
\end{aligned}$$

Since $y=\sqrt{9-x^2}$ implies that $x^2+y^2=3^2$, since we're taking the positive square root, and since $0 \le x \le 3$, the integral gives the area of the quarter of the circle of radius 3 centred at the origin that lies in the first quadrant (*i.e.* where $x \ge 0$ and $y \ge 0$). □

SOLUTION TO **b**. Here's a sketch of the solid, with the original region shaded in:



The volume is about as easy to compute with either the washer or the cylindrical shell method. We'll do it with shells; since we rotated about a vertical line and are using shells, we have to integrate with respect to x. The cylindrical shell at x has radius $r = x - 0 = x$ and height $h = x - 0 = x$, so its area is $2\pi rh = 2\pi xx = 2\pi x^2$. We plug this into the volume formula for shells:

$$V = \int_0^2 2\pi rh\,dx = \int_0^2 2\pi x^2\,dx = 2\pi \frac{x^3}{3}\Big|_0^2 = 2\pi\frac{2^3}{3} - 2\pi\frac{0^3}{3} = 2\pi\frac{8}{3} - 0 = \frac{16}{3}\pi$$

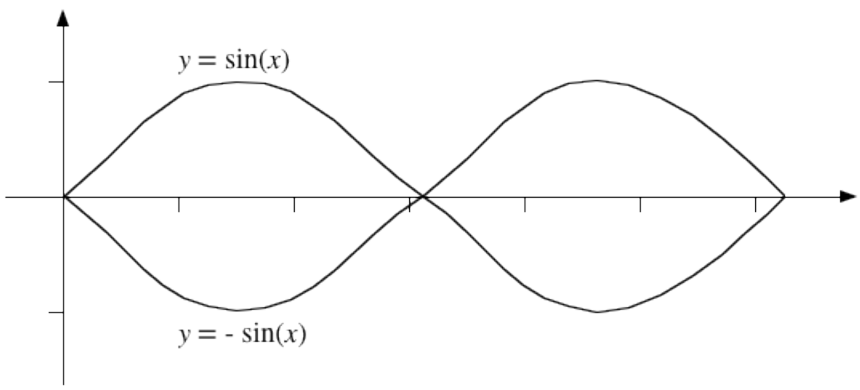
Note that, as always, the limits for the integral come from the original region. □

SOLUTION TO **c**. First, note that $f'(0) = 1 - \int_0^0 f(x)\,dx = 1 - 0 = 1$. Second, note that it follows from the Fundamental Theorem of Calculus that

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\left(1 - \int_0^x f(t)\,dt\right) = 0 - f(x) = -f(x)\,.$$

So, how many functions do you know such that $f''(x) = -f(x)$ and $f'(0) = 1$? Both $\sin(x)$ and $\cos(x)$ meet the first requirement. Since $\frac{d}{dx}\sin(x) = \cos(x)$ and $\cos(0) = 1$, $f(x) = \sin(x)$ does the job. □

SOLUTION TO **d**. Here's a crude sketch:



Note that between 0 and π, $\sin(x) \geq 0$, so $\sin(x) \geq -\sin(x)$, and between π and 2π, $\sin(x) \leq 0$, so $-\sin(x) \geq \sin(x)$. It follows that the area of the region is:

$$\begin{aligned}
A &= \int_0^\pi (\sin(x) - (-\sin(x)))\,dx + \int_\pi^{2\pi} ((-\sin(x)) - \sin(x))\,dx \\
&= \int_0^\pi 2\sin(x)\,dx - \int_\pi^{2\pi} 2\sin(x)\,dx = -2\cos(x)|_0^\pi - (-2\cos(x))|_\pi^{2\pi} \\
&= [-2\cos(\pi) - (-2\cos(0))] - [-2\cos(2\pi) - (-2\cos(\pi))] \\
&= [-2(-1) - (-2\cdot 1)] - [-2\cdot 1 - (-2(-1))] = [2+2] - [-2-2] = 8
\end{aligned}$$ □

Solution to **e**. We plug into the Right-hand Rule formula, namely $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + i\frac{b-a}{n}\right)$, and chug away. In this case $a = 1$, $b = 2$, and $f(x) = x$.

$$\begin{aligned}
\int_1^2 x\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2-1}{n} f\left(1 + i\frac{2-1}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} f\left(1 + \frac{i}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \left[\left(\sum_{i=1}^{n} 1\right) + \left(\sum_{i=1}^{n} \frac{i}{n}\right)\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[n + \frac{1}{n} \sum_{i=1}^{n} i\right] = \lim_{n\to\infty} \frac{1}{n} \left[n + \frac{1}{n} \cdot \frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[n + \frac{n+1}{2}\right] = \lim_{n\to\infty} \frac{1}{n} \left[\frac{3}{2}n + \frac{1}{2}\right] \\
&= \lim_{n\to\infty} \left[\frac{3}{2} + \frac{1}{2n}\right] = \frac{3}{2} + 0 = \frac{3}{2} \quad \square
\end{aligned}$$

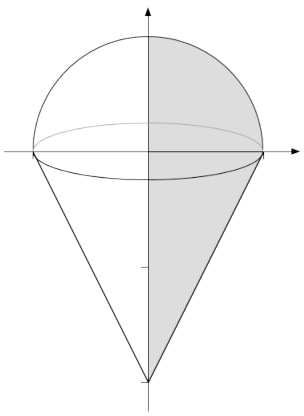
3. The region between $y = \sqrt{1-x^2}$ and $y = 2x - 2$, where $0 \leq x \leq 1$, is rotated about the y-axis to make a solid. Do part **a** and *one* (1) of parts **b** or **c**.

a. Sketch the solid of revolution described above. *[3]*

b. Find the volume of the solid using the disk/washer method. *[9]*

c. Find the volume of the solid using the method of cylindrical shells. *[9]*

SOLUTION TO **a**. Here's a sketch of the solid, with the original region shaded in:



Anyone for ice cream? □

SOLUTION TO **b**. Since we rotated about a vertical line and are using washers, we have to integrate with respect to y. y runs from -2 – the y-intercept of $y = 2x - 2$ – to 1 – the y-intercept of $y = \sqrt{1-x^2}$ – for the given region. The problem is that the outer radius of the washer at y is $R = x = \frac{1}{2}y + 1$ for $-2 \leq y \leq 0$, but is $R = x = \sqrt{1-y^2}$ for $0 \leq y \leq 1$, so we will have to break the integral up accordingly. Note that the inner radius of each washer is $r = 0$ either way, so every washer is actually a disk. We plug all this into the volume formula for the washer method:

$$\begin{aligned}
V &= \int_{-2}^{0} \pi\left(R^2 - r^2\right)\,dy + \int_{0}^{1} \pi\left(R^2 - r^2\right)\,dy \\
&= \pi \int_{-2}^{0} \left(\left(\frac{1}{2}y + 1\right)^2 - 0^2\right)\,dy + \pi \int_{0}^{1} \left(\left(\sqrt{1-y^2}\right)^2 - 0^2\right)\,dy \\
&= \pi \int_{-2}^{0} \left(\frac{1}{4}y^2 + y + 1\right)\,dy + \pi \int_{0}^{1} \left(1 - y^2\right)\,dy \\
&= \pi \left(\frac{1}{4} \cdot \frac{y^3}{3} + \frac{y^2}{2} + y\right)\Bigg|_{-2}^{0} + \pi\left(y - \frac{y^3}{3}\right)\Bigg|_{0}^{1} \\
&= \pi\left(\frac{0^3}{12} + \frac{0^2}{2} + 0\right) - \pi\left(\frac{(-2)^3}{12} + \frac{(-2)^2}{2} + (-2)\right) + \pi\left(1 - \frac{1^3}{3}\right) - \pi\left(0 - \frac{0^3}{3}\right) \\
&= 0 - \pi\left(\frac{-8}{12} + \frac{4}{2} - 2\right) + \pi\frac{2}{3} - 0 = -\pi\frac{-2}{3} + \pi\frac{2}{3} = \frac{4}{3}\pi \quad \square
\end{aligned}$$

SOLUTION TO **c**. Since we rotated about a vertical line and are using shells, we have to integrate with respect to x. The cylindrical shell at x has radius $r = x$ and height $h = \sqrt{1-x^2} - (2x-2) = \sqrt{1-x^2} - 2x + 2$. We plug these into the volume formula for the shell method:

$$V = \int_0^1 2\pi r h \, dx = 2\pi \int_0^1 x \left(\sqrt{1-x^2} - 2x + 2\right) dx$$

$$= 2\pi \int_0^1 x\sqrt{1-x^2}\, dx - 2\pi \int_0^1 2x^2 \, dx + 2\pi \int_0^1 2x \, dx$$

In the first integral, substitute $u = 1 - x^2$, so $du = -2x\, dx$ and $(-1)\, du = 2x\, dx$, and change limits accordingly: $\begin{matrix} x & 0 & 1 \\ u & 1 & 0 \end{matrix}$

$$= \pi \int_1^0 \sqrt{u}(-1)\, du - 4\pi \left. \frac{x^3}{3} \right|_0^1 + 2\pi \left. x^2 \right|_0^1$$

$$= \pi \int_0^1 u^{1/2}\, du - 4\pi \left[\frac{1^3}{3} - \frac{0^3}{3}\right] + 2\pi \left[1^2 - 0^2\right] = \pi \left. \frac{u^{3/2}}{3/2} \right|_0^1 - \frac{4}{3}\pi + 2\pi$$

$$= \pi \left[\frac{2}{3} 1^{3/2} - \frac{2}{3} 0^{3/2}\right] + \frac{2}{3}\pi = \frac{2}{3}\pi + \frac{2}{3}\pi = \frac{4}{3}\pi \quad \square$$

This is the extra page!

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