Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, SUMMER 2011

Solutions to Assignment #4
Inverse hyperbolic trig functions
and an excess of fine print!

The *hyperbolic trigonometric functions,* often just the *hyperbolic functions,* are so named because they relate angles to side length in triangles in the hyperbolic plane*, just as the ordinary trigonometric functions do in triangles in the Euclidean plane. The three principal ones are:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

There are connections between the hyperbolic and the regular trigonometric functions, some of which will become apparent when we study series. Your main task in this assignment will be to work out the inverses of $\sinh(x)$ and $\cosh(x)$.

1. Use `Maple` to graph $\sinh(x)$ and $\cosh(x)$. *[1]*

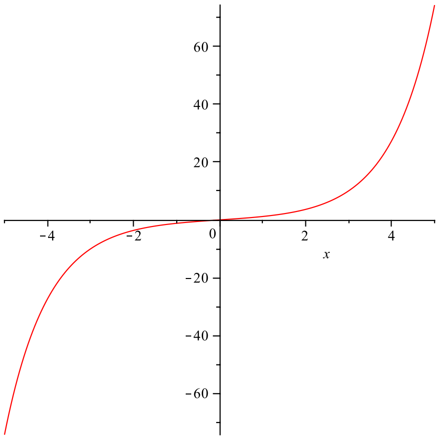
SOLUTION. The commands

```
> plot(sinh(x),x=-5..5);
```

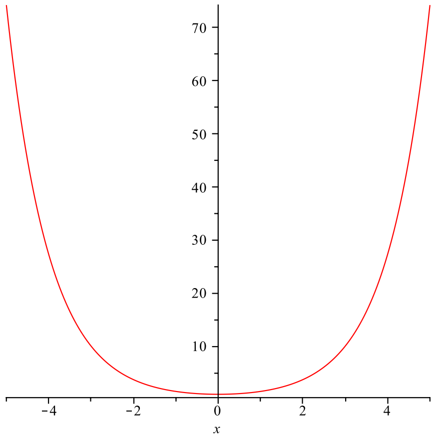
and

```
> plot(cosh(x),x=-5..5);
```

respectively give the graphs



and



. □

* The hyperbolic plane is just like the Euclidean plane except that parallel lines work differently: instead of having just one line parallel to a given line through any point not on the given line, there are infinitely many lines parallel to the given line through any point not on the given line. (One immediate consequence is that the lines through a point parallel to a given line are not parallel to each other.) Hyperbolic geometry, and other non-Euclidean geometries, actually have uses. For one example, the key idea in general relativity is that mass and energy affect the curvature of space, giving it a non-Euclidean geometry.

2. Use your graphs from **1** to determine just how much of each of $\sinh(x)$ and $\cosh(x)$ could be inverted. *[2]*

SOLUTION. The trick is to see how much of each graph can pass the Horizontal Line Test, *i.e.* intersect any horizontal line at most once. (See §1.6 in the text for the Horizontal Line Test.)

For $\sinh(x)$, a look makes it pretty clear that any horizontal line will intersect the graph exactly once; that is, the entire graph passes the Horizontal Line Test. It follows that all of $\sinh(x)$ can be inverted.

For $\cosh(x)$, a look at the graph makes it pretty clear that the horizontal line $y = a$ will intersect that graph not at all if $a < 1$, just once if $a = 1$, and twice if $a > 1$. The largest components of the graph that survive the Horizontal Line Test are the left half of the graph, *i.e.* for $x \leq 0$, and the right half of the graph, *i.e.* for $x \geq 0$. It follows that $\cosh(x)$ can be inverted for $x \leq 0$ or for $x \geq 0$, but not both. □

3. Use `Maple`'s ability to solve equations symbolically to find expressions for the inverses of $\sinh(x)$ and $\cosh(x)$, namely $\text{arcsinh}(x)$ and $\text{arccosh}(x)^{\dagger}$. *[3]*

Note: The basic tool you will need to do **3** is `Maple`'s `solve` command, which has many options and variations. Be warned that while `Maple` is an enormously powerful and flexible tool for doing algebra symbolically, it suffers from a defect of this virtue: it is sometimes very hard for a non-expert user to figure out how to extract a useful result even for a relatively simple problem. Part of **3** boils down to solving a quadratic equation for e^x, and `Maple`'s obsession with generality makes this best done a little indirectly. *Make sure to ask for help if you run into trouble!*

SOLUTION. The general idea is that to find f^{-1} as a function of x, let $f(y) = x$ and then solve for y to get $y = f^{-1}(x)$. Using the definition given above for sinh, this means that we set $\dfrac{e^y - e^{-y}}{2} = x$ and ask `Maple` to solve for y:

```
> solve(x=(exp(y)-exp(-y))/2,y);
```

$$\ln(x + \sqrt{x^2+1}),\ \ln(x - \sqrt{x^2+1})$$

Similarly, using the definition of cosh given above, this means that we set $\dfrac{e^y + e^{-y}}{2} = x$ and ask `Maple` to solve for y:

```
> solve(x=(exp(y)+exp(-y))/2,y);
```

$$\ln(x + \sqrt{x^2-1}),\ \ln(x - \sqrt{x^2-1})$$

The above took a few tries by your instructor, who also gets frustrated with how finicky `Maple` is, too. Here, for example, is part of a previous try for cosh:

```
> a := e^y;
```

$$a := e^y$$

† Our textbook uses the notation $\sinh^{-1}(x)$ and $\cosh^{-1}(x)$, respectively, for the inverses of $\sinh(x)$ and $\cosh(x)$.

```
> solve(x=(a+1/a)/2,a);
Warning, solving for expressions other than names or functions is not recommended.
```

$$x + \sqrt{x^2 - 1}, x - \sqrt{x^2 - 1}$$

```
> solve(e^y = x+sqrt(x^2-1), y);
```

$$\frac{\ln\left(x + \sqrt{x^2 - 1}\right)}{\ln(e)}$$

The $\ln(e)$ in the denominator is there because in many modes `Maple` does not recognize the symbol e as the base of the natural exponential and logarithmic functions. (Hence the use of `exp( )` in the solutions given above, which `Maple` uses for the natural exponential function in classic worksheet mode ...)

A final point is to try to figure out which of the two answers for each of $\operatorname{arcsinh}(x)$ and $\operatorname{arccosh}(x)$ is correct. We'll defer figuring that out until we answer **4** below. □

4. Derive expressions for $\operatorname{arcsinh}(x)$ and $\operatorname{arccosh}(x)$ yourself. (If these are different from what `Maple` gave you for **3**, you may well be correct, but try to explain, if you can, why they amount to the same thing.) *[2]*

SOLUTION. We'll invert sinh first. As in the solution to **3**, we set $x = \dfrac{e^y - e^{-y}}{2}$ and try to solve for y:

$$\begin{aligned}
x = \frac{e^y - e^{-y}}{2} \quad &\implies \quad 2x = e^y - e^{-y} = e^y - \frac{1}{e^y} \\
&\implies \quad 2xe^y = \left(e^y\right)^2 - 1 \\
&\implies \quad \left(e^y\right)^2 - 2xe^y - 1 = 0 \qquad \text{(A quadratic equation in } e^y\text{!)} \\
&\implies \quad e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} \\
&\implies \quad e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1} \\
&\implies \quad y = \ln\left(x + \sqrt{x^2 + 1}\right) \quad \text{or} \quad y = \ln\left(x - \sqrt{x^2 + 1}\right)
\end{aligned}$$

It remains to determine which of the two alternatives is correct. Note that since $x^2 + 1 > x^2$, $\sqrt{x^2 + 1} > |x|$ for all x. It follows that $x + \sqrt{x^2 + 1} > 0$ and $x - \sqrt{x^2 + 1} < 0$ for all x. Since $\ln(t)$ is defined only for $t > 0$, only $y = \ln\left(x + \sqrt{x^2 + 1}\right)$ makes sense among our two possible answers.

Thus $\operatorname{arcsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$.

Now we'll invert cosh. As in the solution to **3**, we set $x = \dfrac{e^y + e^{-y}}{2}$ and try to solve for y:

$$
\begin{aligned}
x = \frac{e^y + e^{-y}}{2} \quad & \Longrightarrow \quad 2x = e^y + e^{-y} = e^y + \frac{1}{e^y} \\
& \Longrightarrow \quad 2xe^y = \left(e^y\right)^2 + 1 \\
& \Longrightarrow \quad \left(e^y\right)^2 - 2xe^y + 1 = 0 \qquad \text{(A quadratic equation in } e^y\text{!)} \\
& \Longrightarrow \quad e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} \\
& \Longrightarrow \quad e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1} \\
& \Longrightarrow \quad y = \ln\left(x + \sqrt{x^2 - 1}\right) \quad \text{or} \quad y = \ln\left(x - \sqrt{x^2 - 1}\right)
\end{aligned}
$$

It remains to determine which of the two alternatives is correct. Note first that $\sqrt{x^2-1}$ only makes sense when $x^2 - 1 \geq 0$, that is, when $x^2 \geq 1$, *i.e.* $x \geq 1$ or $x \leq -1$. Also, since $x^2 - 1 < x^2$, $\sqrt{x^2-1} < |x|$ whenever $\sqrt{x^2-1}$ makes sense. It follows that $x + \sqrt{x^2-1} > 0$ when $x \geq 1$ and $x + \sqrt{x^2-1} < 0$ when $x \leq -1$, while $x - \sqrt{x^2-1} > 0$ when $x \geq 1$ and $x - \sqrt{x^2-1} < 0$ when $x \leq -1$. Since $\ln(t)$ is defined only for $t > 0$, both of our possible answers work when $x \geq 1$. Both are inverses of cosh: $\ln\left(x + \sqrt{x^2-1}\right)$ inverts the right half of cosh and $\ln\left(x - \sqrt{x^2-1}\right)$ inverts the left half of cosh. (See the answer to **2**.)

It is convention to choose to invert the right half of cosh, so $\operatorname{arccosh}(x) = \ln\left(x + \sqrt{x^2-1}\right)$. □

5. Find the derivatives of $\sinh(x)$, $\cosh(x)$, $\operatorname{arcsinh}(x)$, and $\operatorname{arccosh}(x)$. *[2]*

Solution. Just to show that `Maple` is not always excessively finicky, we'll use the `diff` command to get the job done:

```
> diff(sinh(x),x);
```

$$\cosh(x)$$

```
> diff(cosh(x),x);
```

$$\sinh(x)$$

```
> diff(arcsinh(x),x);
```

$$\frac{1}{\sqrt{x^2+1}}$$

```
> diff(arccosh(x),x);
```

$$\frac{1}{\sqrt{x-1}\sqrt{x+1}}$$

Beats doing it by hand! □