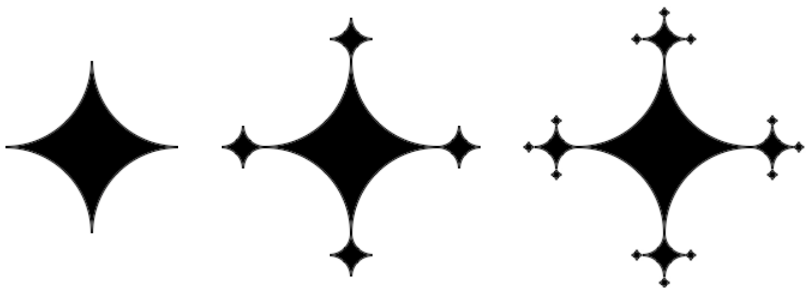
Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, SUMMER 2011

Solutions to Assignment #11
Cross of Squircles

Let's call the shape that you get by removing four mutually tangent quarter-circles with radius $\frac{s}{2}$ from a square with side length s a *squircle**. (See the leftmost shape in the diagram below.)



A single squircle has four points where the quarter-circles that were removed met. Consider the following process:

At step $n = 0$ we have a single squircle for which $s = 2$.

At step $n = 1$, we attach four squircles for which $s = \frac{1}{4} \cdot 2 = \frac{1}{2}$ to the squircle in step 0, attaching one (at one of its points) to each point of the larger squircle. (See the middle shape in the diagram above.) The resulting shape has $3 \cdot 4 = 12$ points (where quarter-circles met) to which nothing is yet attached. Let's call these the *free* points of the shape.

At step $n = 2$, we attach a squircle for which $s = \frac{1}{4} \cdot \frac{1}{2} = \left(\frac{1}{4}\right)^2 \cdot 2 = \frac{1}{8}$ to each of the free points in the shape in step 1. (See the rightmost shape in the diagram above.) The resulting shape has $3 \cdot 12 = 3 \cdot (3 \cdot 4) = 3^2 \cdot 4 = 36$ free points.

At step $n = 3$, we attach a squircle for which $s = \frac{1}{4} \cdot \frac{1}{8} = \left(\frac{1}{4}\right)^3 \cdot 2 = \frac{1}{32}$ to each of these the free points in the shape in step 2. (Draw your own picture!) The resulting shape has $3 \cdot 36 = 3 \cdot \left(3^2 \cdot 4\right) = 3^3 \cdot 4 = 108$ free points.

Repeat for each integer $n > 3$ ...

1. Find formulas for the values of s for the squircles added at step n and for the number of free points of the shape obtained in step n. *[1]*

SOLUTIONS. Denote the value of s for the squircles added at step n by s_n. We know from the description of the process given above that $s_0 = 2 = \left(\frac{1}{4}\right)^0 \cdot 2$, $s_1 = \left(\frac{1}{4}\right)^1 \cdot 2$, $s_2 = \left(\frac{1}{4}\right)^2 \cdot 2$, and $s_3 = \left(\frac{1}{4}\right)^3 \cdot 2$. Continuing this pattern gives $s_n = \left(\frac{1}{4}\right)^n \cdot 2 = \frac{2}{4^n}$.

Denote the number of free points of the shape obtained at step n by p_n. We know from the description of the process given above that $p_0 = 4 = 3^0 \cdot 4$, $p_1 = 3^1 \cdot 4$, $p_2 = 3^2 \cdot 4$, and $p_3 = 3^3 \cdot 4$. Continuing this pattern gives $p_n = 3^n \cdot 4$.

Note that both s_n and p_n are geometric sequences. □

* No doubt this shape already has a name, but I don't know it ...

2. Find a formula for the length of the perimeter (*i.e.* border) of the shape obtained in step n. *[1.5]*

SOLUTIONS. A quarter-circle with radius $\frac{s}{2}$ has perimeter $\frac{1}{4}2\pi\frac{s}{2} = \frac{\pi}{4}s$. It follows that the corresponding squircle has perimeter $4\frac{\pi}{4}s = \pi s$. In particular, at step 0, $s = s_0 = 2$, so the perimeter of the squircle that is the shape at this step is 2π.

Combining the perimeter formula for a squircle obtained above with the formula for s_n obtained in the solution to **1** tells us that the perimeter of each of the squircles added at step n is $\pi s_n = \pi\frac{2}{4^n} = \frac{2\pi}{4^n}$.

At step $n > 0$ we add as many squircles, each with $s = s_n$, to the shape as there were free points at the previous step; from the solution to **1**, this number is $p_{n-1} = 3^{n-1} \cdot 4$. It follows that at each step $n > 0$ we add $p_{n-1} \cdot \pi s_n = 3^{n-1} \cdot 4 \cdot \frac{2\pi}{4^n} = 2\pi\frac{3^{n-1}}{4^{n-1}}$ to the perimeter of the shape at the previous step.

Thus the shape obtained in step n has perimeter:

$$\begin{aligned}
&2\pi + \frac{2\pi 3^{1-1}}{4^{1-1}} + \frac{2\pi 3^{2-1}}{4^{2-1}} + \frac{2\pi 3^{3-1}}{4^{3-1}} + \cdots + \frac{2\pi 3^{n-1}}{4^{n-1}} \\
&= 2\pi + \frac{2\pi 3^0}{4^0} + \frac{2\pi 3^1}{4^1} + \frac{2\pi 3^2}{4^2} + \cdots + \frac{2\pi 3^{n-1}}{4^{n-1}} = 2\pi + \sum_{k=0}^{n-1} 2\pi\frac{3^k}{4^k} \\
&= 2\pi + \sum_{k=0}^{n-1} 2\pi \left(\frac{3}{4}\right)^k = 2\pi + \frac{2\pi\left(1 - \left(\frac{3}{4}\right)^n\right)}{1 - \frac{3}{4}} = 2\pi + \frac{2\pi\left(1 - \left(\frac{3}{4}\right)^n\right)}{\frac{1}{4}} \\
&= 2\pi + 8\pi\left(1 - \left(\frac{3}{4}\right)^n\right) = 10\pi - 8\pi\left(\frac{3}{4}\right)^n \qquad \square
\end{aligned}$$

3. Find a formula for the area of the shape obtained in step n. *[1.5]*

SOLUTIONS. A square with side length s has area s^2 and quarter-circle with radius $\frac{s}{2}$ has area $\frac{1}{4}\pi\left(\frac{s}{2}\right)^2 = \frac{\pi}{4 \cdot 2^2}s^2 = \frac{\pi}{16}s^2$. It follows that the corresponding squircle has area $s^2 - 4\frac{\pi}{16}s^2 = \left(1 - \frac{\pi}{4}\right)s^2$. In particular, at step 0, $s = s_0 = 2$, so the perimeter of the squircle that is the shape at this step is $\left(1 - \frac{\pi}{4}\right)2^2 = 4 - \pi$.

Combining the area formula for a squircle obtained above with the formula for s_n obtained in the solution to **1** tells us that the area of each of the squircles added at step $n > 0$ is $\left(1 - \frac{\pi}{4}\right)s_n^2 = \left(1 - \frac{\pi}{4}\right)\left(\frac{2}{4^n}\right)^2 = \left(1 - \frac{\pi}{4}\right) \cdot \frac{2^2}{4^{2n}} = \left(1 - \frac{\pi}{4}\right)\frac{1}{4^{2n-1}}$.

At step $n > 0$ we add as many squircles, each with $s = s_n$, to the shape as there were free points at the previous step; from the solution to **1**, this number is $p_{n-1} = 3^{n-1} \cdot 4$. It follows that at each step $n > 0$ we add $p_{n-1} \cdot \left(1 - \frac{\pi}{4}\right)s_n^2 = 3^{n-1} \cdot \left(1 - \frac{\pi}{4}\right)\frac{1}{4^{2n-1}} =$

$\left(1-\frac{\pi}{4}\right)\frac{3^{n-1}4}{4^{2n-1}} = \left(1-\frac{\pi}{4}\right)\frac{3^{n-1}}{4^{2n-2}} = \left(1-\frac{\pi}{4}\right)\frac{3^{n-1}}{16^{n-1}}$ to the area of the shape at the previous step.

Thus the shape obtained at step n has area:

$$\begin{aligned}
&4\left(1-\frac{\pi}{4}\right) + \left(1-\frac{\pi}{4}\right)\frac{3^{1-1}}{16^{1-1}} + \left(1-\frac{\pi}{4}\right)\frac{3^{2-1}}{16^{2-1}} + \cdots + \left(1-\frac{\pi}{4}\right)\frac{3^{n-1}}{16^{n-1}} \\
&= \left(1-\frac{\pi}{4}\right)\left[4 + \frac{3^0}{16^0} + \frac{3^1}{16^1} + \frac{3^2}{16^2} + \frac{3^{n-1}}{16^{n-1}}\right] \\
&= \left(1-\frac{\pi}{4}\right)\left[4 + \left(\frac{3}{16}\right)^0 + \left(\frac{3}{16}\right)^1 + \left(\frac{3}{16}\right)^2 + \left(\frac{3}{16}\right)^{n-1}\right] \\
&= \left(1-\frac{\pi}{4}\right)\left[4 + \frac{1-\left(\frac{3}{16}\right)^n}{1-\frac{3}{16}}\right] = \left(1-\frac{\pi}{4}\right)\left[4 + \frac{1-\left(\frac{3}{16}\right)^n}{\frac{13}{16}}\right] \\
&= \left(1-\frac{\pi}{4}\right)\left[4 + \frac{16}{13} - \left(\frac{3}{16}\right)^n\right] = \left(1-\frac{\pi}{4}\right)\left[\frac{68}{13} - \left(\frac{3}{16}\right)^{n-1}\right] \quad \square
\end{aligned}$$

4. Compute the length of the perimeter of the shape obtained after infinitely many steps of the process. *[3]*

SOLUTIONS. This amounts to taking the limit of the formula obtained in **2**:

$$\lim_{n\to\infty}\left[10\pi - 8\pi\left(\frac{3}{4}\right)^n\right] = [10\pi - 8\pi\cdot 0] = 10\pi\,,$$

since $\left(\frac{3}{4}\right)^n \to 0$ as $n\to\infty$ (because $\left|\frac{3}{4}\right| < 1$). It follows that the perimeter of the shape obtained after infinitely many steps of the process has length 10π. $\square$

5. Compute the area of the shape obtained after infinitely many steps of the process. *[3]*

SOLUTIONS. This amounts to taking the limit of the formula obtained in **4**:

$$\lim_{n\to\infty}\left(1-\frac{\pi}{4}\right)\left[\frac{68}{13} - \left(\frac{3}{16}\right)^{n-1}\right] = \left(1-\frac{\pi}{4}\right)\left[\frac{68}{13} - 0\right] = \frac{68}{13}\left(1-\frac{\pi}{4}\right)\,,$$

since $\left(\frac{3}{16}\right)^{n-1} \to 0$ as $n\to\infty$ (because $\left|\frac{3}{16}\right| < 1$). It follows that the area of the shape obtained after infinitely many steps of the process is $\frac{68}{13}\left(1-\frac{\pi}{4}\right)$. $\square$