Mathematics 1100Y – Calculus I: Calculus of one variable
Trent University, Summer 2011
Solutions to the Final Examination

Time: 09:00–12:00, on Wednesday, 3 August, 2011. *Brought to you by* Стефан.
Instructions: Show all your work and justify all your answers. *If in doubt,* **ask!**
Aids: Calculator; two (2) aid sheets [all 12 sides]; one (1) brain [may be caffeinated].

Part I. Do *all* three (3) of **1–3**.

1. Compute $\frac{dy}{dx}$ as best you can in any *three* (3) of **a**–**f**. *[15 = 3 × 5 each]*

a. $x = e^{x+y}$ **b.** $y = \int_0^{-x} te^t\,dt$ **c.** $y = x^2\ln(x)$

d. $y = \frac{x}{\cos(x)}$ **e.** $y = \sec^2(\arctan(x))$ **f.** $y = \sin(e^x)$

Solution *i* to **a**. We will use implicit differentiation. (The alternative is to solve for y first, which isn't too hard in this case; see solution *ii* below.) Differentiating both sides of $x = e^{x+y}$ gives

$$1 = \frac{dx}{dx} = \frac{d}{dx}e^{x+y} = e^{x+y}\cdot\frac{d}{dx}(x+y) = e^{x+y}\cdot\left(\frac{dx}{dx}+\frac{dy}{dx}\right)$$
$$= e^{x+y}\cdot\left(1+\frac{dy}{dx}\right) = e^{x+y}+e^{x+y}\frac{dy}{dx},$$

from which it follows that

$$e^{x+y}\frac{dy}{dx} = 1-e^{x+y} \implies \frac{dy}{dx} = \frac{1}{e^{x+y}}-\frac{e^{x+y}}{e^{x+y}} = e^{-(x+y)}-1. \qquad \square$$

Solution *ii* to **a**. We will solve for y as a function of x first, and then differentiate.

$$x = e^{x+y} = e^xe^y \implies e^y = \frac{x}{e^x} = xe^{-x}$$
$$\implies y = \ln(e^y) = \ln\left(xe^{-x}\right) = \ln(x)+\ln\left(e^{-x}\right) = \ln(x)-x\ln(e) = \ln(x)-x$$

It follows that
$$\frac{dy}{dx} = \frac{d}{dx}(\ln(x)-x) = \frac{1}{x}-1.$$

(If you plug $y=\ln(x)-x$ into $\frac{dy}{dx} = e^{-(x+y)}-1$ it works out to $\frac{1}{x}-1$ too.) $\square$

Solution to **b**. We will use the Chain Rule and the Fundamental Theorem of Calculus. Let $u=-x$; then $y = \int_0^u te^t\,dt$ and so

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \left(\frac{d}{du}\int_0^u te^t\,dt\right)\left(\frac{d}{dx}(-x)\right) = ue^u(-1) = (-x)e^{-x}(-1) = xe^{-x}.$$

One could also do this by computing the definite integral first using integration by parts, and then differentiating. $\square$

SOLUTION TO **c**. We will do this one using the Product Rule as the main tool:

$$\frac{dy}{dx} = \frac{d}{dx}x^2\ln(x) = \left(\frac{d}{dx}x^2\right) \cdot \ln(x) + x^2 \cdot \left(\frac{d}{dx}\ln(x)\right) = 2x\ln(x) + x^2\frac{1}{x} = 2x\ln(x) + x \quad \square$$

SOLUTION TO **d**. We will do this one using the Quotient Rule as the main tool:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x}{\cos(x)}\right) = \frac{\frac{dx}{dx} \cdot \cos(x) - x \cdot \frac{d}{dx}\cos(x)}{\cos^2(x)} \\ &= \frac{1\cos(x) - x\,(-\sin(x))}{\cos^2(x)} = \frac{\cos(x) + x\sin(x)}{\cos^2(x)}\end{aligned}$$

Given the multitude of trig identities, which could be applied before or after differentiating, there are lots of equivalent ways of writing the answer. For one nice example, $\frac{dy}{dx} = \sec(x) + x\sec(x)\tan(x)$. $\square$

SOLUTION TO **e**. This can be done using the Chain Rule and following up with a trig identity or two, but it's easier if one simplifies y using a trig identity or two first:

$$y = \sec^2(\arctan(x)) = 1 + \tan^2(\arctan(x)) = 1 + (\tan(\arctan(x)))^2 = 1 + x^2$$

It then follows that $\frac{dy}{dx} = \frac{d}{dx}\left(1+x^2\right) = 0 + 2x = 2x$. $\square$

SOLUTION TO **f**. There is no avoiding the Chain Rule in this one:

$$\frac{dy}{dx} = \frac{d}{dx}\sin(e^x) = \cos(e^x) \cdot \frac{d}{dx}e^x = \cos(e^x) \cdot e^x = e^x\cos(e^x) \quad \square$$

2. Evaluate any *three* (3) of the integrals **a**–**f**. *[15 = 3 × 5 each]*

a. $\displaystyle\int \frac{2x}{\sqrt{4-x^2}}\,dx$ **b.** $\displaystyle\int_0^{\pi/2} \sin(z)\cos(z)\,dz$ **c.** $\displaystyle\int x^2\ln(x)\,dx$

d. $\displaystyle\int_{-\infty}^{\ln(3)} e^s\,ds$ **e.** $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx$ **f.** $\displaystyle\int_1^2 \frac{1}{w^2+w}\,dw$

SOLUTION TO **a**. This can be done using a trig substitution, namely $x = 2\sin(\theta)$, but it's faster to use the substitution $u = 4 - x^2$, so that $du = -2x\,dx$ and $2x\,dx = (-1)\,du$:

$$\begin{aligned}\int \frac{2x}{\sqrt{4-x^2}}\,dx &= \int \frac{1}{\sqrt{u}}(-1)\,du = -\int u^{-1/2}\,du \\ &= -\frac{u^{1/2}}{1/2} + C = -2\sqrt{u} + C = -2\sqrt{4-x^2} + C \quad \square\end{aligned}$$

SOLUTION TO **b**. This is probably easiest using the substitution $u = \sin(z)$, so $du = \cos(z)\,dz$ and $\begin{matrix} z & 0 & \pi/2 \\ u & 0 & 1 \end{matrix}$.

$$\int_0^{\pi/2} \sin(z)\cos(z)\,dz = \int_0^1 u\,du = \left.\frac{u^2}{2}\right|_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2} - 0 = \frac{1}{2} \quad \square$$

SOLUTION TO **c**. We will use integration by parts with $u = \ln(x)$ and $v' = x^2$, so $u' = \frac{1}{x}$ and $v = \frac{x^3}{3}$:

$$\begin{aligned} \int x^2 \ln(x)\,dx &= \frac{x^3}{3}\ln(x) - \int \frac{1}{x} \cdot \frac{x^3}{3}\,dx \\ &= \frac{x^3}{3}\ln(x) - \int \frac{x^2}{3}\,dx = \frac{x^3}{3}\ln(x) - \frac{x^3}{9} + C \quad \square \end{aligned}$$

SOLUTION TO **d**. This is an improper integral, so we need to set up and compute a limit:

$$\begin{aligned} \int_{-\infty}^{\ln(3)} e^s\,ds &= \lim_{t\to-\infty} \int_t^{\ln(3)} e^s\,ds = \lim_{t\to-\infty} e^s\big|_t^{\ln(3)} \\ &= \lim_{t\to-\infty} \left(e^{\ln(3)} - e^t\right) = \lim_{t\to-\infty} \left(3 - e^t\right) = 3 - 0 = 3\,, \end{aligned}$$

since $e^t \to 0$ as $t \to -\infty$. $\square$

SOLUTION TO **e**. We will use the trig substitution $x = \tan(\theta)$, so $dx = \sec^2(\theta)\,d\theta$.

$$\begin{aligned} \int \frac{1}{\sqrt{1+x^2}}\,dx &= \int \frac{1}{\sqrt{1+\tan^2(\theta)}}\sec^2(\theta)\,d\theta = \int \frac{1}{\sqrt{\sec^2(\theta)}}\sec^2(\theta)\,d\theta \\ &= \int \frac{1}{\sec(\theta)}\sec^2(\theta)\,d\theta = \int \sec(\theta)\,d\theta = \ln\left(\tan(\theta) + \sec(\theta)\right) + C \\ &= \ln\left(x + \sqrt{1+x^2}\right) + C \end{aligned}$$

Note that $\sec(\theta) = \sqrt{sec^2(\theta)} = \sqrt{1+\tan^2(\theta)} = \sqrt{1+x^2}$ if $x = \tan(\theta)$. $\square$

SOLUTION TO **f**. This integral requires the use of partial fractions. First, if

$$\frac{1}{w^2+w} = \frac{1}{w(w+1)} = \frac{A}{w} + \frac{B}{w+1} = \frac{A(w+1)+Bw}{w(w+1)} = \frac{(A+B)w + A}{w^2+w}\,,$$

then, comparing numerators, we must have $A + B = 0$ and $A = 1$, so $B = -1$. It follows that

$$\begin{aligned} \int_1^2 \frac{1}{w^2+w}\,dw &= \int_1^2 \left(\frac{1}{w} + \frac{-1}{w+1}\right) dw = \int_1^2 \frac{1}{w}\,dw - \int_1^2 \frac{1}{w+1}\,dw \\ &= \ln(w)\big|_1^2 - \ln(w+1)\big|_1^2 = [\ln(2) - \ln(1)] - [\ln(2+1) - \ln(1+1)] \\ &= [\ln(2) - 0] - [\ln(3) - \ln(2)] = \ln(2) - \ln(3) + \ln(2) \\ &= 2\ln(2) - \ln(3) \end{aligned}$$

Those who feel compelled to simplify further may do so:

$$2\ln(2) - \ln(3) = \ln\left(2^2\right) - \ln(3) = \ln(4) - \ln(3) = \ln\left(\frac{4}{3}\right) \quad \square$$

3. Do any *five* (5) of **a**–**i**. *[25 = 5 × 5 ea.]*

a. Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n^2}{3^n}$ converges absolutely, converges conditionally, or diverges.

b. Why must the arc-length of $y = \arctan(x)$, $0 \le x \le 13$, be less than $13 + \frac{\pi}{2}$?

c. Find a power series equal to $f(x) = \dfrac{x}{1+x}$ (when the series converges) without using Taylor's formula.

d. Find the area of the region between the origin and the polar curve $r = \frac{\pi}{2} + \theta$, where $0 \le \theta \le \pi$.

e. Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{n}{2^n} x^n$.

f. Use the limit definition of the derivative to compute $f'(0)$ for $f(x) = 2x - 1$.

g. Compute the area of the surface obtained by rotating the the curve $y = \dfrac{x^2}{2}$, where $0 \le x \le \sqrt{3}$, about the y-axis.

h. Use the Right-hand Rule to compute the definite integral $\displaystyle\int_0^3 (x+1)\,dx$.

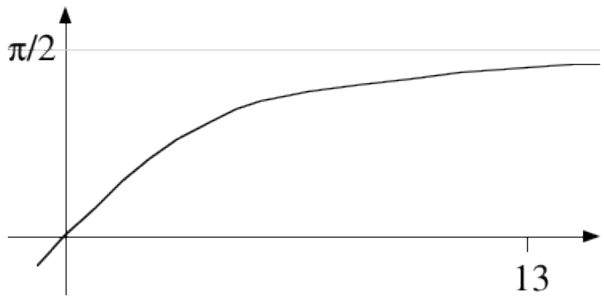
i. Use the $\varepsilon - \delta$ definition of limits to verify that $\displaystyle\lim_{x\to 2}(x+1) = 3$.

Solution to **a**. We will apply the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(n+1)^2}{3^{n+1}}}{\frac{(-1)^n n^2}{3^n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)^2}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n^2}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-1)\left(n^2+2n+1\right)}{3n^2}\right| = \lim_{n\to\infty}\frac{n^2+2n+1}{3n^2} \\
&= \lim_{n\to\infty}\left(\frac{n^2}{3n^2} + \frac{2n}{3n^2} + \frac{1}{3n^2}\right) = \lim_{n\to\infty}\left(\frac{1}{3} + \frac{2}{3n} + \frac{1}{3n^2}\right) = \frac{1}{3} + 0 + 0 = \frac{1}{3}
\end{aligned}$$

Since $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3} < 1$, the given series converges absolutely by the Ratio Test. □

SOLUTION TO **b**. The easiest way to see – and explain – why the arc-length of $y = \arctan(x)$, $0 \le x \le 13$, must be less than $13 + \frac{\pi}{2}$ is to draw a picture; even a crude sketch will suffice:



(Note that since arctan(13) is pretty close to $\frac{\pi}{2}$, the point $\left(13, \frac{\pi}{2}\right)$ is pretty close to the point $(13, \arctan(13))$.) It should be pretty clear from the picture that going from the origin by way of the y-axis to the point $\left(0, \frac{\pi}{2}\right)$ and then on to the point $\left(13, \frac{\pi}{2}\right)$ by way of the line $y = \frac{\pi}{2}$ is a longer trip than going from the origin to $(13, \arctan(13))$ by way of the curve $y = \arctan(x)$. It follows that $\frac{\pi}{2} + 13$ is greater than the arc-length of $y = \arctan(x)$ for $0 \le x \le 13$. □

SOLUTION TO **c**. Using the formula for the sum of a geometric series in reverse,

$$\begin{aligned} f(x) &= \frac{x}{1+x} = \frac{x}{1-(-x)} = \sum_{n=0}^{\infty} x(-x)^n = \sum_{n=0}^{\infty} x(-1)^n x^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^{n+1} = x - x^2 + x^3 - x^4 + x^5 - \cdots \quad \square \end{aligned}$$

SOLUTION TO **d**. We plug the polar curve $r = \frac{\pi}{2} + \theta$, $0 \le \theta \le \pi$, into the area formula in polar coordinates:

$$\begin{aligned} \int_0^\pi \frac{1}{2} r^2 \, d\theta &= \int_0^\pi \frac{1}{2}\left(\frac{\pi}{2} + \theta\right)^2 d\theta = \frac{1}{2}\int_0^\pi \left(\left(\frac{\pi}{2}\right)^2 + 2\frac{\pi}{2}\theta + \theta^2\right) d\theta \\ &= \frac{1}{2}\left(\frac{\pi^2}{4}\theta + \frac{\pi}{2}\theta^2 + \frac{1}{3}\theta^3\right)\Bigg|_0^\pi \\ &= \frac{1}{2}\left(\frac{\pi^2}{4}\pi + \frac{\pi}{2}\pi^2 + \frac{1}{3}\pi^3\right) - \frac{1}{2}\left(\frac{\pi^2}{4}0 + \frac{\pi}{2}0^2 + \frac{1}{3}0^3\right) = \frac{13}{24}\pi^3 \quad \square \end{aligned}$$

SOLUTION TO **e**. We will first find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{n}{2^n} x^n$

using the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{n+1}{2^{n+1}}x^{n+1}}{\frac{n}{2^n}x^n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{2^{n+1}}x\cdot\frac{2^n}{n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{x}{2}\right|$$
$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{|x|}{2} = (1+0)\frac{|x|}{2} = \frac{|x|}{2}$$

It follows by the Ratio Test that the series converges absolutely when $\frac{|x|}{2} < 1$, *i.e.* when $-2 < x < 2$, and diverges when $\frac{|x|}{2} > 1$, *i.e.* when $x < -2$ or $x > 2$. Thus the radius of convergence of the given power series is $R = 2$.

It remains to determine what happens at the endpoints of the interval, namely at $x = -2$ and $x = 2$. Since

$$\lim_{n\to\infty}\left|\frac{n}{2^n}(-2)^n\right| = \lim_{n\to\infty}|(-1)^n n| = \lim_{n\to\infty} n = \infty \neq 0$$
$$\text{and}\quad \lim_{n\to\infty}\left|\frac{n}{2^n}2^n\right| = \lim_{n\to\infty} n = \infty \neq 0\,,$$

the Divergence Test tells us that the series diverges for both $x = -2$ and $x = 2$. The interval of convergence is therefore $(-2, 2)$. □

SOLUTION TO **f**. We will plug $f(x) = 2x - 1$ into the limit definition of the derivative to compute $f'(0)$:

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{(2(0+h)-1)-(2\cdot 0-1)}{h}$$
$$= \lim_{h\to 0}\frac{2h-1-(-1)}{h} = \lim_{h\to 0}\frac{2h}{h} = \lim_{h\to 0} 2 = 2 \quad \square$$

SOLUTION TO **g**. We will plug the curve $y = \frac{x^2}{2}$, $0 \leq x \leq \sqrt{3}$, into the formula for the area of a surface of revolution, $\int 2\pi r\, ds$. First, note that since we are rotating the curve about the y-axis, $r = x - 0 = x$. Second, since $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2}{2}\right) = \frac{1}{2}2x = x$, we have that $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+x^2}\,dx$. Thus

$$\text{SA} = \int 2\pi r\, ds = \int_0^{\sqrt{3}} 2\pi x\sqrt{1+x^2}\,dx \quad \text{Let } u = 1+x^2\text{, so } du = 2x\,dx \text{ and } \begin{matrix} x & 0 & \sqrt{3} \\ u & 1 & 4 \end{matrix}.$$
$$= \pi\int_1^4 \sqrt{u}\,du = \pi\int_1^4 u^{1/2}\,du = \frac{2}{3}u^{3/2}\Big|_1^4 = \frac{2}{3}\left(4^{3/2}-1^{3/2}\right) = \frac{2}{3}(8-1) = \frac{14}{3} \quad \square$$

Solution to **h**. We plug the definite integral $\int_0^3 (x+1)\,dx$ into the Right-hand Rule formula $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + \frac{b-a}{n} i\right)$ and chug away:

$$\begin{aligned}
\int_0^3 (x+1)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3-0}{n}\left[\left(0+\frac{3-0}{n}i\right)+1\right] = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\frac{3}{n}i+1\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{3}{n}i+1\right] = \lim_{n\to\infty} \frac{3}{n}\left[\left(\sum_{i=1}^{n}\frac{3}{n}i\right)+\left(\sum_{i=1}^{n}1\right)\right] \\
&= \lim_{n\to\infty} \frac{3}{n}\left[\frac{3}{n}\left(\sum_{i=1}^{n} i\right)+n\right] = \lim_{n\to\infty} \frac{3}{n}\left[\frac{3}{n}\cdot\frac{n(n+1)}{2}+n\right] \\
&= \lim_{n\to\infty} \frac{3}{n}\left[\frac{3}{2}(n+1)+n\right] = \lim_{n\to\infty} \frac{3}{n}\left[\frac{5}{2}n+\frac{3}{2}\right] = \lim_{n\to\infty}\left[\frac{15}{2}+\frac{9}{2n}\right] = \frac{15}{2}
\end{aligned}$$

since $\frac{9}{2n} \to 0$ as $n \to \infty$. □

Solution to **i**. We need to show that for any $\varepsilon > 0$, there is a corresponding $\delta > 0$ such that whenever $|x-2| < \delta$, we have $|(x+1)-3| < \varepsilon$.

Suppose, then, that $\varepsilon > 0$. As usual, we will reverse-engineer the corresponding $\delta > 0$:

$$|(x+1)-3| < \varepsilon \iff |x-2| < \varepsilon\,,$$

so $\delta = \varepsilon$ will do the job. (One step of reverse-engineering is as easy as it gets!)

Thus $\lim_{x\to 2}(x+1) = 3$ by the $\varepsilon - \delta$ definition of limits. □

Part II. Do any *three* (3) of **4**–**8**.

4. Find the domain, all maximum, minimum, and inflection points, and all vertical and horizontal asymptotes of $f(x) = \frac{x^2}{x^2+1}$, and sketch its graph. *[15]*

Solution. We'll run through the usual check list of items and then sketch the graph:

i. (Domain) Since $x^2+1 \geq 1 > 0$ for all x, the denominator is never 0. It follows that the rational function $f(x) = \frac{x^2}{x^2+1}$ is defined (and continuous and differentiable, too) for all x, *i.e.* its domain is $\mathbb{R} = (-\infty, \infty)$.

ii. (Intercepts) $f(0) = \frac{0^2}{0^2+1} = 0$, so the y-intercept is the origin, *i.e.* at $y = 0$. On the other hand, $f(x) = \frac{x^2}{x^2+1} = 0$ is only possible if the numerator $x^2 = 0$, *i.e.* $x = 0$. It follows that the origin is also the only x-intercept.

iii. (Vertical asymptotes) Since $f(x)$ is defined and continuous for all x, it cannot have any vertical asymptotes.

iv. (Horizontal asymptotes) We compute the relevant limits to check for horizontal asymptotes:

$$\lim_{x\to-\infty}\frac{x^2}{x^2+1}=\lim_{x\to-\infty}\frac{x^2}{x^2+1}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}=\lim_{x\to-\infty}\frac{1}{1+\frac{1}{x^2}}=\frac{1}{1+0}=1$$
$$\lim_{x\to+\infty}\frac{x^2}{x^2+1}=\lim_{x\to+\infty}\frac{x^2}{x^2+1}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}=\lim_{x\to+\infty}\frac{1}{1+\frac{1}{x^2}}=\frac{1}{1+0}=1$$

(Note that $\frac{1}{x^2}\to 0$ both as $x\to-\infty$ and as $x\to+\infty$.) It follows that $f(x)$ has the horizontal asymptote $y=1$ in both directions.

v. (Maxima and minima) First, with a little help from the Quotient Rule,

$$f'(x)=\frac{d}{dx}\left(\frac{x^2}{x^2+1}\right)=\frac{\left(\frac{d}{dx}x^2\right)\cdot\left(x^2+1\right)-x^2\cdot\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2}$$
$$=\frac{2x\cdot\left(x^2+1\right)-x^2\cdot 2x}{\left(x^2+1\right)^2}=\frac{2x^3+2x-2x^3}{\left(x^2+1\right)^2}=\frac{2x}{\left(x^2+1\right)^2}\,.$$

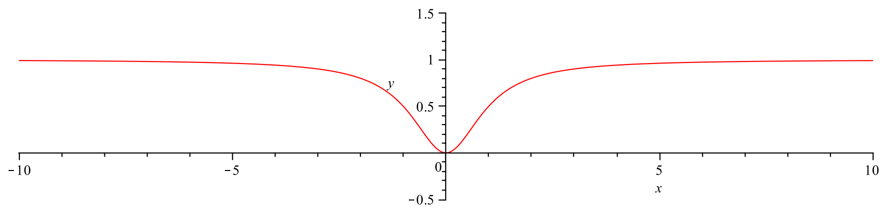
Second, note that $f'(x)$ is also a rational function that is defined and continuous for all x, using reasoning very similar to that used in i above. This means that the only kind of critical point that can occur is the sort where $f'(x)=0$. $f'(0)=\dfrac{2x}{\left(x^2+1\right)^2}=0$ can only occur when the numerator, $2x$, is 0, which happens only when $x=0$. Moreover, since the denominator, $\left(x^2+1\right)^2\geq 1>0$ for all x (and $2>0$, too) $f'(x)<0$ when $x<0$ and $f'(x)>0$ when $x>0$. We can summarize this information and its effect on $f(x)$ with the usual sort of table:

x	$(-\infty,0)$	0	$(0,+\infty)$
$f'(x)$	$-$	0	$+$
$f(x)$	$\downarrow$	min	$\uparrow$

Note that $x=0$ gives a local (and absolute) minimum point for $f(x)$ and that $f(x)$ has no maximum point. (This makes particular sense if you do *iv* above a little more carefully and notice that $f(x)$ approaches the horizontal asymptote from below in both directions.)

vi. (Graph) We cheat ever so slightly and let `Maple` do the drawing:

```
> plot(x^2/(x^2+1),x=-10..10,y=-0.5..1.5);
```



Whew! □

5. Do both of **a** and **b**.

a. Verify that $\displaystyle\int \sqrt{x^2-1}\,dx = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln\left(x+\sqrt{x^2-1}\right) + C$. *[7]*

b. Find the arc-length of $y = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln\left(x+\sqrt{x^2-1}\right)$ for $1 \le x \le 3$. *[8]*

SOLUTION TO **a**. We can compute the indefinite integral using the trig substitution $x = \sec(\theta)$, so $dx = \sec(\theta)\tan(\theta)\,d\theta$, but it's often easier to differentiate the antiderivative and check that the result is equal to the integrand. Trying this here

$$\begin{aligned}
&\frac{d}{dx}\left[\frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln\left(x+\sqrt{x^2-1}\right) + C\right] \\
&= \frac{1}{2}\frac{d}{dx}\left[x\sqrt{x^2-1}\right] - \frac{1}{2}\frac{d}{dx}\ln\left(x+\sqrt{x^2-1}\right) + \frac{d}{dx}C \\
&= \frac{1}{2}\left[\frac{dx}{dx}\cdot\sqrt{x^2-1} + x\cdot\frac{d}{dx}\sqrt{x^2-1}\right] - \frac{1}{2}\frac{1}{x+\sqrt{x^2-1}}\cdot\frac{d}{dx}\left(x+\sqrt{x^2-1}\right) + 0 \\
&= \frac{1}{2}\left[1\cdot\sqrt{x^2-1} + x\cdot\frac{1}{2\sqrt{x^2-1}}\cdot\frac{d}{dx}\left(x^2-1\right)\right] \\
&\qquad - \frac{1}{2}\frac{1}{x+\sqrt{x^2-1}}\cdot\left(1+\frac{1}{2\sqrt{x^2-1}}\cdot\frac{d}{dx}\left(x^2-1\right)\right) \\
&= \frac{1}{2}\left[\sqrt{x^2-1} + \frac{x}{2\sqrt{x^2-1}}\cdot 2x\right] - \frac{1}{2}\frac{1}{x+\sqrt{x^2-1}}\cdot\left(1+\frac{1}{2\sqrt{x^2-1}}\cdot 2x\right) \\
&= \frac{1}{2}\left[\sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}}\right] - \frac{1}{2}\frac{1}{x+\sqrt{x^2-1}}\cdot\left(1+\frac{x}{\sqrt{x^2-1}}\right) \\
&= \frac{1}{2}\left[\sqrt{x^2-1}\cdot\frac{\sqrt{x^2-1}}{\sqrt{x^2-1}} + \frac{x^2}{\sqrt{x^2-1}}\right] - \frac{1}{2}\frac{1}{x+\sqrt{x^2-1}}\cdot\left(\frac{\sqrt{x^2-1}}{\sqrt{x^2-1}}+\frac{x}{\sqrt{x^2-1}}\right) \\
&= \frac{1}{2}\frac{\left(x^2-1\right)+x^2}{\sqrt{x^2-1}} - \frac{1}{2}\frac{1}{x+\sqrt{x^2-1}}\cdot\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}} \\
&= \frac{1}{2}\frac{2x^2-1}{\sqrt{x^2-1}} - \frac{1}{2}\frac{\sqrt{x^2-1}+x}{x+\sqrt{x^2-1}}\cdot\frac{1}{\sqrt{x^2-1}} \\
&= \frac{1}{2}\frac{2x^2-1}{\sqrt{x^2-1}} - \frac{1}{2}1\cdot\frac{1}{\sqrt{x^2-1}} = \frac{1}{2}\frac{2x^2-1-1}{\sqrt{x^2-1}} = \frac{1}{2}\frac{2\left(x^2-1\right)}{\sqrt{x^2-1}} = \frac{x^2-1}{\sqrt{x^2-1}} = \sqrt{x^2-1}
\end{aligned}$$

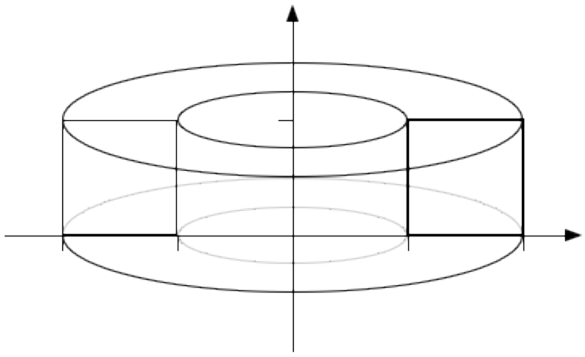
... makes for an algebra-fest of epic proportions. Maybe it would have been easier to integrate ... :-) □

SOLUTION TO **b**. It follows from **a** and the Fundamental Theorem of Calculus that $\frac{dy}{dx} = \sqrt{x^2 - 1}$. Plugging this into the arc-length formula gives:

$$\begin{aligned} \text{arc-length} &= \int_1^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^3 \sqrt{1 + \left(\sqrt{x^2-1}\right)^2}\, dx = \int_1^3 \sqrt{1 + x^2 - 1}\, dx \\ &= \int_1^3 \sqrt{x^2}\, dx = \int_1^3 x\, dx = \left.\frac{x^2}{2}\right|_1^3 = \frac{3^2}{2} - \frac{1^2}{2} = \frac{9-1}{2} = \frac{8}{2} = 4 \quad \square \end{aligned}$$

6. Sketch the solid obtained by rotating the square with corners at $(1,0)$, $(1,1)$, $(2,0)$, and $(2,1)$ about the y-axis and find its volume and surface area. *[15]*

SOLUTION. Here is sketch of the solid:



Note that the original region has as its borders (pieces of) the vertical lines $x = 1$ and $x = 2$ and the horizontal lines $y = 0$ and $y = 1$, *i.e.* the region consists of all points (x, y) with $1 \le x \le 2$ and $0 \le y \le 1$.

We can compute the volume of the solid in several easy ways:

i. (Shells) Note that since we revolved the original region about the y-axis and are using shells, we will have to integrate with respect to x. The cylindrical shell at x, for some $1 \le x \le 2$, has radius $r = x - 0 = x$ and height $h = 1 - 0 = 1$. Thus the volume of the solid is:

$$V = \int_1^2 2\pi r h\, dx = \int_1^2 2\pi x \cdot 1\, dx = 2\pi \frac{1}{2} x^2 \Big|_1^2 = \pi x^2\Big|_1^2 = \pi\left(2^2 - 1^2\right) = 3\pi$$

ii. (Washers) Note that since we revolved the original region about the y-axis and are using washers, we will have to integrate with respect to y. The washer at y, for some $0 \le y \le 1$, has outer radius $R = 2 - 0 = 2$ and inner radius $r = 1 - 0 = 1$. (Note that the washers are all identical ...) Thus the volume of the solid is:

$$V = \int_0^1 \pi\left(R^2 - r^2\right) dy = \int_0^1 \pi\left(2^2 - 1^2\right) dy = \int_0^1 3\pi\, dy = 3\pi y\big|_0^1 = 3\pi(1-0) = 3\pi$$

iii. (Look, Ma! No calculus!) The solid is a cylinder of radius $r = 2$ and height $h = 1$ (and hence volume $\pi r^2 h = 4\pi$) with a cylinder of radius $r = 1$ and height $h = 1$ (and hence volume $\pi r^2 h = \pi$) removed from it. It follows that the solid has volume $4\pi - \pi = 3\pi$.

Any correct method, correctly and completely worked out, would do, of course. :-)

It remains to find the surface area of the solid. The complication is that the surface of the solid consists of four distinct pieces: the upper and lower faces of the solid, which are both washers with outside radius $R = 2$ and inside radius $r = 1$, the outside face, which is a cylinder of radius $R = 2$ and height $h = 1$, and the inside face (*i.e.* the hole in the middle), which is a cylinder of radius $r = 1$ and height $h = 1$. While the area of each of these can be computed pretty quickly as the area of a surface of revolution, it is pointless to work so hard when we should have formulas for the areas of these objects at our fingertips from our knowledge of the washer and shell methods for computing volume:

The areas of the upper and lower faces are each $\pi\left(R^2 - r^2\right) = \pi\left(2^2 - 1^2\right) = 3\pi$, the area of the outside face is $\pi R^2 h = \pi 2^2 1 = 4\pi$, and the area of the inside face is $\pi r^2 h = \pi 1^2 1 = \pi$. The total surface area of the solid is therefore $2 \cdot 3\pi + 4\pi + \pi = 11\pi$. □

7. Do all three (3) of **a**–**c**.

a. Use Taylor's formula to find the Taylor series at 0 of $f(x) = \ln(x+1)$. *[7]*

b. Determine the radius and interval of convergence of this Taylor series. *[4]*

c. Use your answer to part **a** to find the Taylor series at 0 of $\dfrac{1}{x+1}$ without using Taylor's formula. *[4]*

SOLUTION TO **a**. We first differentiate and evaluate away to figure out what $f^{(n)}(0)$ must be for each n. $f^{(0)}(x) = f(x) = \ln(x+1)$, so $f^{(0)}(0) = \ln(0+1) = 0$; $f^{(1)}(x) = f'(x) = \frac{d}{dx}\ln(x+1) = \frac{1}{x+1} \cdot \frac{d}{dx}(x+1) = \frac{1}{x+1}1 = \frac{1}{x+1}$, so $f^{(1)}(0) = \frac{1}{0+1} = 1$; $f^{(2)}(x) = f''(x) = \frac{d}{dx}\frac{1}{x+1} = \frac{-1}{(x+1)^2} \cdot \frac{d}{dx}(x+1) = \frac{-1}{(x+1)^2}1 = \frac{-1}{(x+1)^2}$, so $f^{(2)}(0) = \frac{-1}{(0+1)^2} = -1$; $f^{(3)}(x) = f'''(x) = \frac{d}{dx}\frac{-1}{(x+1)^2} = \frac{(-1)(-2)}{(x+1)^3} \cdot \frac{d}{dx}(x+1) = \frac{(-1)^2 2}{(x+1)^3}1 = \frac{(-1)^2 2}{(x+1)^3}$, so $f^{(3)}(0) = \frac{2}{(0+1)^3} = (-1)^2 2$; and so on:

n	0	1	2	3	4	5	6	$\cdots$
$f^{(n)}(x)$	$\ln(x+1)$	$\frac{1}{x+1}$	$\frac{-1}{(x+1)^2}$	$\frac{(-1)^2 2}{(x+1)^3}$	$\frac{(-1)^3 6}{(x+1)^4}$	$\frac{(-1)^4 24}{(x+1)^5}$	$\frac{(-1)^5 120}{(x+1)^6}$	$\cdots$
$f^{(n)}(0)$	0	1	-1	$(-1)^2 2$	$(-1)^3 6$	$(-1)^4 24$	$(-1)^5 120$	$\cdots$

A little reflection on this pattern shows us that at stage $n > 0$, $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(x+1)^n}$ and so $f^{(n)}(0) = \frac{(-1)^{n-1}(n-1)!}{(0+1)^n} = (-1)^{n-1}(n-1)!$.

Applying Taylor's formula, and noting that $n = 0$ is the exception to the pattern noted above, the Taylor series at 0 of $f(x) = \ln(x+1)$ is therefore:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \frac{f^{(0)}(0)}{0!}x^0 + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \frac{0}{1}1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}x^n$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \quad \square$$

Solution to **b**. To find the radius of convergence of the power series obtained in **a**, we use the Ratio Test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{(n+1)-1}}{n+1}x^{n+1}}{\frac{(-1)^{n-1}}{n}x^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n}{(-1)^{n-1}}\cdot\frac{n}{n=1}\cdot\frac{x^{n+1}}{x^n}\right| \\ &= \lim_{n\to\infty}\left|(-1)\frac{n}{n+1}x\right| = \lim_{n\to\infty}\frac{n}{n+1}|x| = |x|\lim_{n\to\infty}\frac{n}{n+1} \\ &= |x|\lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{1/n}{1/n} = |x|\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = |x|\frac{1}{1+0} = |x|\end{aligned}$$

It follows that the series converges absolutely when $|x|<1$ and diverges when $|x|>1$, so the radius of convergence is $R=1$.

To find the interval of convergence, we have to determine whether the series converges or diverges at each of $x=-1$ and $x=1$. That is, we have to determine whether each of the series

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(-1)^n = \sum_{n=1}^{\infty}\frac{-1}{n} = -1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\cdots \\ \text{and}\quad &\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}1^n = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\end{aligned}$$

converges or not. The first diverges: $-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\cdots = (-1)\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)$, so it is a non-zero multiple of the harmonic series, which diverges. (We showed this in class using the Integral Test, though it is even quicker to use the p-Test.) The second converges: it is just the alternating harmonic series, which converges conditionally. (We showed this in class using the Alternating Series Test.) It follows that the interval of convergence is $(-1,1)\cup\{1\}=(-1,1]$. □

Solution to **c**. Since $\frac{d}{dx}\ln(x+1) = \frac{1}{x+1}$ (as noted in the solution to **a** above), we can get the Taylor series at 0 of $f'(x)=\frac{1}{x+1}$ by differentiating the Taylor series at 0 of $f(x)=\ln(x+1)$ term-by-term:

$$\begin{aligned}&\frac{d}{dx}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots\right) \\ &= \frac{d}{dx}x-\frac{d}{dx}\left(\frac{x^2}{2}\right)+\frac{d}{dx}\left(\frac{x^3}{3}\right)-\frac{d}{dx}\left(\frac{x^4}{4}\right)+\frac{d}{dx}\left(\frac{x^5}{5}\right)-\cdots \\ &= 1-\frac{2x}{2}+\frac{3x^2}{3}-\frac{4x^3}{4}+\frac{5x^4}{5}-\cdots \\ &= 1-x+x^2-x^3+x^4-\cdots\end{aligned}$$

This is the geometric series with first term $a=1$ and common ratio $r=-x$. □

8. A spherical balloon is being inflated at a rate of 1 m^3/s. How is its surface area changing at the instant that its volume is 36 m^3? *[15]*

[Recall that a sphere of radius r has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$.]

SOLUTION. We need to relate $1 = \frac{dV}{dt}$ to $\frac{dA}{dt}$, where A is the surface area of the balloon.

First, observe that

$$\frac{dA}{dt} = \frac{d}{dt}4\pi r^2 = \frac{d}{dr}4\pi r^2 \cdot \frac{dr}{dt} = 8\pi r \cdot \frac{dr}{dt}\,.$$

This means that we will need to know $\dfrac{dr}{dt}$ at the instant in question.

Second, we have

$$1 = \frac{dV}{dt} = \frac{d}{dt}\frac{4}{3}\pi r^3 = \frac{d}{dr}\frac{4}{3}\pi r^3 \cdot \frac{dr}{dt} = \frac{4}{3}\pi 3r^2 \cdot \frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}\,,$$

so $\dfrac{dr}{dt} = \dfrac{1}{4\pi r^2}$.

It follows that $\dfrac{dA}{dt} = 8\pi r \cdot \dfrac{1}{4\pi r^2} = \dfrac{2}{r}$.

Third, it still remains to determine what the value of r is at the instant in question. Since $V = \dfrac{4}{3}\pi r^3 = 36$ at the instant in question, $r = \left(\dfrac{36}{\frac{4}{3}\pi}\right)^{1/3} = \left(\dfrac{27}{\pi}\right)^{1/3} = \dfrac{3}{\pi^{1/3}}$.

Thus, at the instant that the volume is 36 m^3, the surface area is changing at a rate of $\dfrac{dA}{dt} = \dfrac{2}{\frac{3}{\pi^{1/3}}} = \dfrac{2}{3}\pi^{1/3}\ m^2/s$. □

[Total = 100]

Part MMXI - Bonus problems.

13. Show that $\ln(\sec(x) - \tan(x)) = -\ln(\sec(x) + \tan(x))$. *[2]*

SOLUTION. Here goes:

$$\begin{aligned}\ln(\sec(x) - \tan(x)) &= \ln([\sec(x) - \tan(x)] \cdot 1) = \ln\left([\sec(x) - \tan(x)] \cdot \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}\right) \\ &= \ln\left(\frac{\sec^2(x) - \tan^2(x)}{\sec(x) + \tan(x)}\right) = \ln\left(\frac{(1 + \tan^2(x)) - \tan^2(x)}{\sec(x) + \tan(x)}\right) \\ &= \ln\left(\frac{1}{\sec(x) + \tan(x)}\right) = \ln\left([\sec(x) + \tan(x)]^{-1}\right) \\ &= (-1)\ln(\sec(x) + \tan(x)) = -\ln(\sec(x) + \tan(x))\end{aligned}$$

Note that the key trick is the same one we used in class to compute $\int \sec(x)\,dx$. □

41. Write an original poem touching on calculus or mathematics in general. *[2]*

SOLUTION. Write your own! □

I HOPE THAT YOU HAD SOME FUN WITH THIS!

GET SOME REST NOW ...