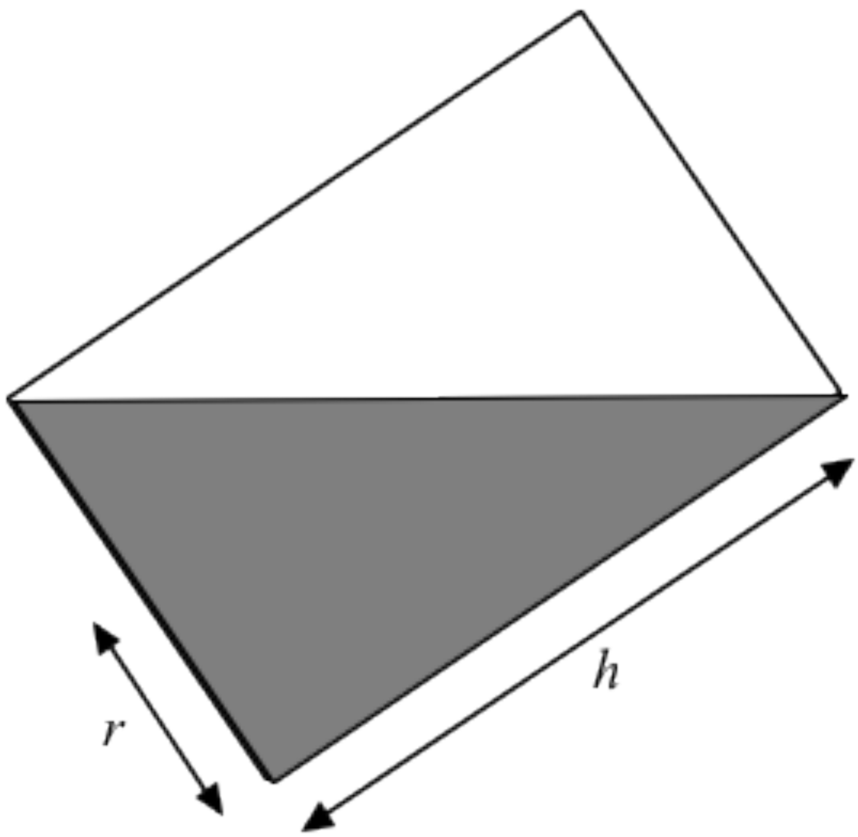
Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2010

Solutions to Assignment #6
Glass half-full?

A cylindrical glass of (interior) radius r and height h is filled with water and then tilted, causing some of the water to pour out, until the remaining water in the glass just barely covers the base of the glass. Here's a side view:



1. Find the volume of the water remaining in the glass without using calculus. *[2]*

SOLUTION. The volume of the glass, using the formula for the volume of a cylinder, is $\pi r^2 h$. In the situation in question, the volume of the water remaining in the glass is exactly half of this, *i.e.* $\frac{1}{2}\pi r^2 h$. To see this note that the empty part of the glass is exactly the same shape as the full part. (Each can be made to coincide with the other by rotating the glass end-over-end 180°.) ■

2. Find the volume of the water remaining in the glass using calculus. *[8]*

SOLUTION. We'll use the idea of integrating the areas of suitable cross-sections to find the desired volume. The problem is to identify cross-sections whose areas we can readily compute. In this case there are at least four more-or-less reasonable choices:

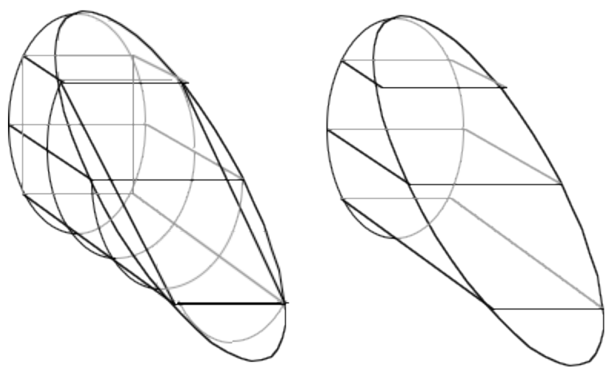
i. cross-sections parallel to the base of the glass, which are disks with (varying amounts of) the top chopped off,

ii. cross-sections perpendicular to both the base of the glass and the surface of the water, which are (almost all) trapezoids,

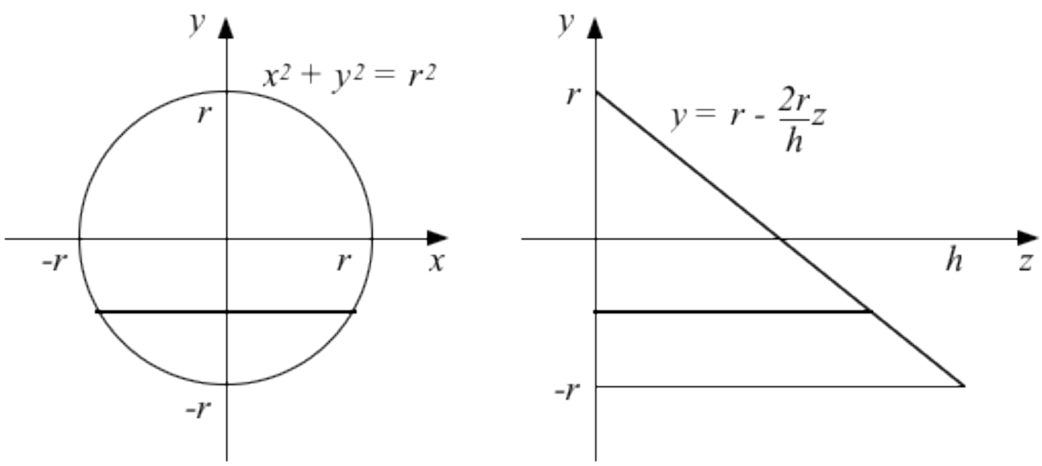
iii. cross-sections perpendicular to the base of the glass and perpendicular to the cross-sections in *ii* above, which are (almost all) rectangles, and

iv. cross-sections parallel to the surface of the water, which are ellipses with (varying amounts of) one end chopped off.

See the left figure below for an example or three of types *i*–*iii* of these cross-sections. (Type *iv* are left to your imagination and artistic skills …) The right figure shows just the type *iii* cross-sections.



Which to use? We'll go with choice *iii*, as it is easy to find the areas of rectangles once their dimensions are known and it is not too hard to determine the dimensions of the rectangles in this case. (*i*, *ii*, and *iv* all turn out to have some additional degree of complication by comparison; *ii* is probably the next easiest overall.) To determine the dimensions of the rectangles, we impose axes on end- and side-views of the solid in question as in the figures below:



Note that for this to be useful, the y-axes in the two views must coincide. We leave it to you work out that the equations in the figures are correct.

From the figures, the rectangle at height y has width $2|x| = 2\sqrt{r^2 - y^2}$ and length $z = \frac{h}{2r}(r - y)$, and hence area $2|x|z = 2\sqrt{r^2 - y^2} \cdot \frac{h}{2r}(r - y) = \frac{h}{r}(r - y)\sqrt{r^2 - y^2}$. The range of possible y-values is obviously $-r \leq y \leq r$. Thus the volume of the solid region is given by:

$$\int_{-r}^{r} \frac{h}{r}(r-y)\sqrt{r^2-y^2}\,dy = \int_{-r}^{r} \frac{h}{r} r\sqrt{r^2-y^2}\,dy - \int_{-r}^{r} \frac{h}{r} y\sqrt{r^2-y^2}\,dy$$

$$= h\int_{-r}^{r} \sqrt{r^2-y^2}\,dy - \frac{h}{r}\int_{-r}^{r} y\sqrt{r^2-y^2}\,dy$$

The former integral we can do with the trig substitution $y = r\sin(\theta)$, so $dy = r\cos(\theta)\,d\theta$ and $\begin{array}{ccc} y & -r & r \\ \theta & -\pi/2 & \pi/2 \end{array}$. For the latter integral we use the substitution $u = r^2 - y^2$, so $du = -2y\,dy$ (and so $-\frac{1}{2}\,du = y\,dy$) and $\begin{array}{ccc} y & -r & r \\ u & 0 & 0 \end{array}$.

$$= h\int_{-\pi/2}^{\pi/2} \sqrt{r^2 - r^2\sin^2(\theta)}\, r\cos(\theta)\,d\theta - \frac{h}{r}\int_{0}^{0} \sqrt{u}\left(-\frac{1}{2}\right)du$$

$$= h\int_{-\pi/2}^{\pi/2} r\cos(\theta) r\cos(\theta)\,d\theta - \frac{h}{r}0$$

Question for you: Why does the latter integral $= 0$?

$$= hr^2\int_{-\pi/2}^{\pi/2} \cos^2(\theta)\,d\theta$$

$$= hr^2\int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(1 + \cos(2\theta)\right)\,d\theta$$

$$= \frac{1}{2}hr^2\int_{-\pi/2}^{\pi/2} 1\,d\theta + \frac{1}{2}hr^2\int_{-\pi/2}^{\pi/2} \cos(2\theta)\,d\theta$$

$$= \frac{1}{2}hr^2\theta\Big|_{-\pi/2}^{\pi/2} + 0$$

Question for you: Why does the latter integral $= 0$?

$$= \frac{1}{2}hr^2\frac{\pi}{2} - \frac{1}{2}hr^2\left(-\frac{\pi}{2}\right) = \frac{1}{2}\pi hr^2,$$

as desired. Whew! ■