Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2010

Quiz Solutions

Quiz #1 *Wednesday, 12 May, 2010. [10 minutes]*

1. Suppose the graph of $y = x^2$ is stretched vertically by a factor of 3, and then shifted by 2 units to the right and 1 unit down. Find the formula of the parabola with this curve as its graph. *[5]*
2. Use the Limit Laws to evaluate $\lim_{x \to 0} \frac{x^2 - 1}{x^2 + 1}$. *[5]*

SOLUTION TO 1. To stretch the graph of $y = x^2$ vertically by a factor of 3, we simply multiply the output by 3 to get $y = 3x^2$. Shifting the graph by 2 units to the right corresponds to replacing x by $x - 2$ to get $y = 3(x-2)^2$. To shift the graph down by 1, we just subtract 1 to get $y = 3(x-2)^2 - 1$. It follows that the formula of the desired parabola is $y = 3(x-2)^2 - 1 = 3x^2 - 12x + 11$. ■

SOLUTION TO 2. Here goes – you should be able to identify the Limit Law(s) used at each step for yourself pretty readily:

$$\lim_{x \to 0} \frac{x^2 - 1}{x^2 + 1} = \frac{\lim_{x \to 0} \left(x^2 - 1\right)}{\lim_{x \to 0} \left(x^2 + 1\right)} = \frac{\left(\lim_{x \to 0} x^2\right) - \left(\lim_{x \to 0} 1\right)}{\left(\lim_{x \to 0} x^2\right) + \left(\lim_{x \to 0} 1\right)} = \frac{0^2 - 1}{0^2 + 1} = \frac{-1}{1} = -1 \qquad \blacksquare$$

Quiz #2 *Monday, 17 May, 2010. [12 minutes]*

Do *one* (1) of the following two questions.

1. Find all the vertical and horizontal asymptotes of $f(x) = \frac{x}{x-1}$ and give a rough sketch of its graph. *[10]*
2. Use the ε–δ definition of limits to verify that $\lim_{x \to 1} (3x - 1) = 2$. *[10]*

SOLUTION TO 1. To find the horizontal asymptotes, we need only compute the limits of $f(x)$ as x tends to $+\infty$ and $-\infty$, respectively, and see what happens:

$$\lim_{x \to +\infty} \frac{x}{x-1} = \lim_{x \to +\infty} \frac{x}{x-1} \cdot \frac{1/x}{1/x} = \lim_{x \to +\infty} \frac{1}{1 - 1/x} = \frac{1}{1-0} = 1^+$$
$$\lim_{x \to -\infty} \frac{x}{x-1} = \lim_{x \to -\infty} \frac{x}{x-1} \cdot \frac{1/x}{1/x} = \lim_{x \to -\infty} \frac{1}{1 - 1/x} = \frac{1}{1+0} = 1^-$$

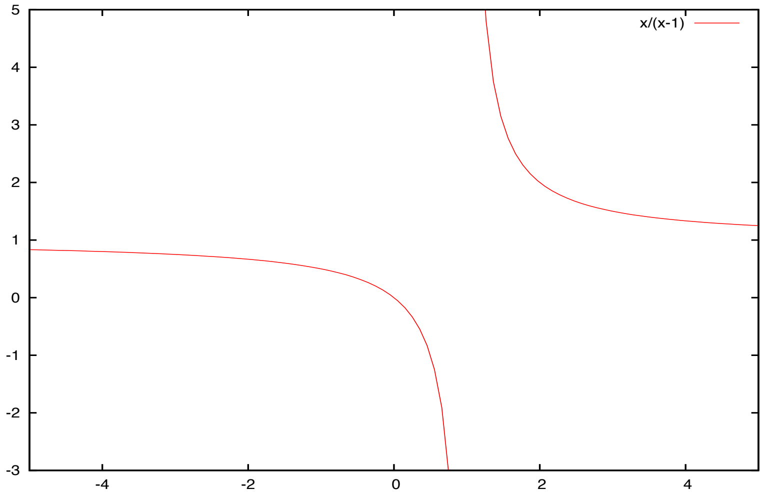
Thus $f(x)$ has $y = 1$ as an asymptote in both directions. Note that it approaches this asymptote from above when $x \to +\infty$ and from below when $x \to -\infty$.

To find any vertical asymptotes, we first need to find the points at which $f(x)$ is undefined. Since the expression $\frac{x}{x-1}$ makes sense for any x unless the denominator is 0, *i.e.* when $x = 1$. To actually check for vertical asymptotes at, we now compute the limits $f(x)$ as x tends to 1 from the left and the right, respectively, and see what happens:

$$\lim_{x \to 1^-} \frac{x}{x-1} = \frac{1}{1^- - 1} = \frac{1}{0^-} = -\infty$$
$$\lim_{x \to 1^+} \frac{x}{x-1} = \frac{1}{1^+ - 1} = \frac{1}{0^+} = +\infty$$

Thus $f(x)$ has a vertical asymptote at $x = 1$; $f(x)$ shoots down to $-\infty$ as x approaches 1 from the left and $f(x)$ shoots up to $+\infty$ as x approaches 1 from the right.

To graph $f(x)$ it's also convenient to note that $f(0) = \frac{0}{0-1} = 0$, so $(0, 0)$ is both and x- and y-intercept. Here's a graph of $f(x) = \dfrac{x}{x-1}$:



This graph was created using `Yacas` ("Yet Another Computer Algebra System"), a free program that can do much of what `Maple` and `Mathematica` can. ■

Solution to 2. We need to show that for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x - 1| < \delta$, then $|(3x - 1) - 2| < \varepsilon$.

Suppose, then, that an $\varepsilon > 0$ is given. We will find a corresponding $\delta > 0$ by reverse-engineering $|(3x - 1) - 2| < \varepsilon$ to look a much as possible like $|x - 1| < \delta$:

$$\begin{aligned} |(3x-1)-2| < \varepsilon &\iff |3x-3| < \varepsilon \\ &\iff |3(x-1)| < \varepsilon \\ &\iff 3\,|x-1| < \varepsilon \\ &\iff |x-1| < \frac{\varepsilon}{3} \end{aligned}$$

Since the steps are all reversible, it follows that $\delta = \frac{\varepsilon}{3}$ does the job: if $|x - 1| < \delta$, then $|(3x - 1) - 2| < \varepsilon$.

It follows by the ε–δ definition of limits that $\lim_{x \to 1}(3x - 1) = 2$. ■

Quiz #3 *Wednesday, 19 May, 2010. [10 minutes]*

1. Compute the derivative of $f(x) = \dfrac{x^2 - 2x}{x - 1}$. *[5]*
2. Compute the derivative of $g(x) = \arctan(e^x)$. *[5]*

SOLUTION TO 1. Our main tool here is the Quotient Rule:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{x^2 - 2x}{x - 1}\right) \\ &= \frac{\left[\frac{d}{dx}\left(x^2 - 2x\right)\right] \cdot (x-1) - \left(x^2 - 2x\right) \cdot \left[\frac{d}{dx}(x-1)\right]}{(x-1)^2} \\ &= \frac{(2x-2) \cdot (x-1) - \left(x^2 - 2x\right) \cdot 1}{(x-1)^2} \\ &= \frac{2x^2 - 4x + 2 - x^2 + 2x}{(x-1)^2} \\ &= \frac{x^2 - 2x + 2}{(x-1)^2} \end{aligned}$$

For those determined to simplify further, one could rewrite $x^2 - 2x + 2$ as $(x-1)^2 + 1$ and take it from there, but one doesn't really gain much by this. ■

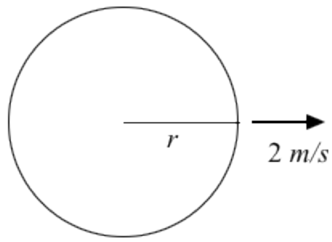
SOLUTION TO 2. The main tool for this one is the Chain Rule:

$$\begin{aligned} g'(x) &= \frac{d}{dx} \arctan(e^x) \\ &= \arctan'(e^x) \cdot \frac{d}{dx} e^x \\ &= \frac{1}{1 + (e^x)^2} \cdot e^x \\ &= \frac{e^x}{1 + e^{2x}} \end{aligned}$$

Those who really want to could also rewrite this as $\frac{1}{e^{-x} + e^x}$. ■

Quiz #4 *Wednesday, 26 May, 2010. [12 minutes]*

1. Use logarithmic differentiation to compute the derivative of $g(x) = x^x$. *[5]*
2. A pebble is dropped into a still pond, creating a circular ripple that moves outward from its centre at 2 m/s. How is the area enclosed by the ripple changing at the instant that the radius of the ripple is 3 m? *[5]*



(*Just in case:* The area of a circle with radius r is πr^2.)

SOLUTION TO 1. We will take the derivative of $\ln(g(x))$ and then solve for $g'(x)$. On the one hand,

$$\begin{aligned}\frac{d}{dx}\ln(g(x)) = \frac{d}{dx}\ln(x^x) &= \frac{d}{dx}(x\ln(x)) \qquad \text{[Using properties of logarithms.]}\\ &= \left(\frac{d}{dx}x\right)\cdot\ln(x) + x\cdot\frac{d}{dx}\ln(x) \qquad \text{[Using the Product Rule.]}\\ &= 1\cdot\ln(x) + x\cdot\frac{1}{x} = \ln(x)+1\,,\end{aligned}$$

and on the other hand, it follows from the Chain Rule that

$$\frac{d}{dx}\ln(g(x)) = \frac{1}{g(x)}\cdot g'(x)\,.$$

Hence

$$g'(x) = g(x)\cdot\frac{d}{dx}\ln(g(x)) = x^x\cdot(\ln(x)+1)\ .$$

(Those with a taste for perversity may rearrange this as

$$g'(x) = x^x\cdot(\ln(x)+1) = x^x\ln(x) + x^x = \ln\left(x^{(x^x)}\right) + x^x\,,$$

but that's just a little sickening ... :-) ■

SOLUTION TO 2. The area of a circle of radius r is $A = \pi r^2$; we wish to know $\frac{dA}{dt}$ at the instant in question. Using the Chain Rule,

$$\frac{dA}{dt} = \frac{d}{dt}\pi r^2 = \pi\left(\frac{d}{dr}r^2\right)\cdot\frac{dr}{dt} = \pi\cdot 2r\frac{dr}{dt} = 2\pi r\frac{dr}{dt}\,.$$

Plugging in the given values, that $r = 3$ at the instant we're interested in and that $\frac{dr}{dt} = 2$, we thus get:

$$\frac{dA}{dt} = 2\pi\cdot 3\cdot 2 = 12\pi$$

Thus the area enclosed by the ripple is changing at a rate of 12π m^2/s at the instant in question. ■

Quiz #5 *Monday, 31 May, 2010. [15 minutes]*

1. Let $f(x) = \dfrac{x}{x^2+1}$. Find the domain and all the intercepts, vertical and horizontal asymptotes, and local maxima and minima of $f(x)$, and sketch its graph using this information. *[10]*

SOLUTION. Here goes!

i. (*Domain.*) The expression $\frac{x}{x^2+1}$ makes sense for all x. (Note that the denominator is always ≥ 1.) Thus the domain of $f(x)$ is all of $\mathbb{R}$. $\Box$

ii. (*Intercepts.*) $f(0) = 0$, so $(0,0)$ is the only x-intercept. Since $\frac{x}{x^2+1}$ can only equal 0 if the numerator, x, is 0, $(0,0)$ is also the only y-intercept. $\Box$

iii. (*Vertical asymptotes.*) Since there are no points at which $f(x)$ is not defined and continuous, it has no vertical asymptotes. $\Box$

iv. (*Horizontal asymptotes.*) Since

$$\lim_{x\to+\infty} \frac{x}{x^2+1} = \lim_{x\to+\infty} \frac{x}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{1+\frac{1}{x^2}} = \frac{0}{1+0} = 0 \quad \text{and}$$
$$\lim_{x\to-\infty} \frac{x}{x^2+1} = \lim_{x\to-\infty} \frac{x}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{\frac{1}{x}}{1+\frac{1}{x^2}} = \frac{0}{1+0} = 1\,,$$

$f(x)$ has the horizontal asymptote $y = 0$ in both directions. Note that as $x \to +\infty$, $\frac{x}{x^2+1} > 0$, since the numerator and denominator are both positive when $x > 0$. Similarly, as $x \to -\infty$, $\frac{x}{x^2+1} < 0$, since the numerator is negative and the denominator is positive when $x < 0$. It follows that $f(x)$ approaches the horizontal asymptote $y = 0$ from above when heading out to $+\infty$, and from below when heading out to $-\infty$. $\Box$

v. (*Local maxima and minima.*) We'll need the derivative of $f(x)$, which we compute using the Quotient Rule:

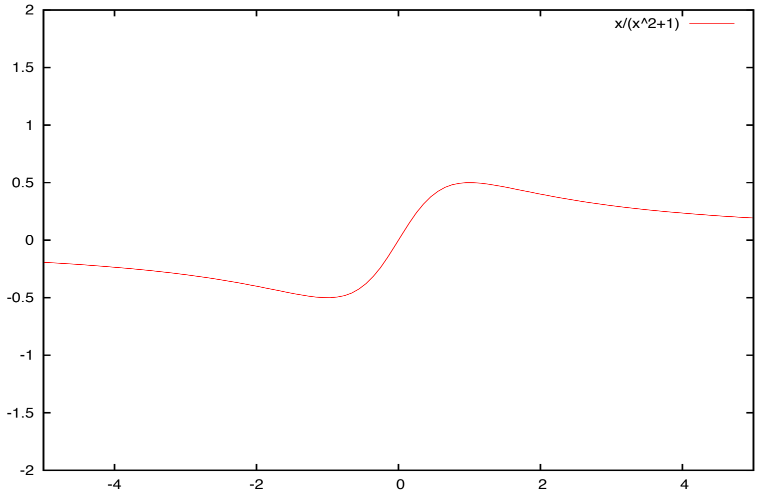
$$f'(x) = \frac{\frac{dx}{dx} \cdot \left(x^2+1\right) - x \cdot \frac{d}{dx}(x^2+1)}{\left(x^2+1\right)^2} = \frac{1 \cdot \left(x^2+1\right) - x \cdot 2x}{\left(x^2+1\right)^2} = \frac{1-x^2}{\left(x^2+1\right)^2}$$

To fond the critical points, observe that $f'(x) = 0$ exactly when $1 - x^2 = 0$. Since $1 - x^2 = (1+x)(1-x)$, this means that $f'(x) = 0$ for $x = -1$ and $x + 1$. We construct the usual table to determine if these are local maxima, minima, or neither:

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
$f'(x)$	<0	0	>0	0	<0
$f(x)$	$\downarrow$	$-\frac{1}{2}$	$\uparrow$	$\frac{1}{2}$	$\downarrow$

It follows that $f(-1) - -\frac{1}{2}$ is a local minimum and $f(1) = \frac{1}{2}$ is a local maximum of $f(x)$. Note that $f'(x)$, like $f(x)$, is defined and continuous everywhere, so the critical points are all we need to check when looking for local maxima and minima. $\Box$

vi. (*Graph.*) Here's a graph of $f(x) = \dfrac{x}{x^2+1}$:



This graph was created using `Yacas` ("Yet Another Computer Algebra System"), a free program that can do much of what `Maple` and `Mathematica` can. □

That's all, folks! ■

Quiz #6 *Wednesday, 2 June, 2010. [10 minutes]*

1. Use the Left-Hand Rule to compute $\int_0^1 (x+1)\,dx$, the area between the line $y = x+1$ and the x-axis for $0 \le x \le 1$. *[10]*

   Hint: You may need the formula $\displaystyle\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$.

SOLUTION. We plug $f(x) = x+1$, $a = 0$, and $b = 1$ into the Left-Hand Rule formula and compute the resulting limit:

$$\begin{aligned}
\int_0^1 (x+1)\,dx &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{b-a}{n} f\left(a + i\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1-0}{n} f\left(0 + i\frac{1-0}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{n} f\left(\frac{i}{n}\right) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{n}\left(\frac{i}{n}+1\right) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(\frac{i}{n^2}+\frac{1}{n}\right) \\
&= \left(\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{i}{n^2}\right) + \left(\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{n}\right) = \left(\lim_{n\to\infty} \frac{1}{n^2} \sum_{i=0}^{n-1} i\right) + \left(\lim_{n\to\infty} n\frac{1}{n}\right) \\
&= \left(\lim_{n\to\infty} \frac{1}{n^2} \frac{(n-1)n}{2}\right) + \left(\lim_{n\to\infty} 1\right) = \left(\lim_{n\to\infty} \frac{n-1}{2n}\right) + 1 \\
&= \left(\lim_{n\to\infty} \frac{n-1}{2n} \cdot \frac{1/n}{1/n}\right) + 1 = \left(\lim_{n\to\infty} \frac{1-\frac{1}{n}}{2}\right) + 1 = \frac{1-0}{2} + 1 = \frac{3}{2} \qquad \blacksquare
\end{aligned}$$

Quiz #7 *Monday, 7 June, 2010. [10 minutes]*

1. Compute $\displaystyle\int_0^2 \left(x^2 - 2x + 1\right)\, dx$. *[10]*

SOLUTION. Here goes, in entirely excessive detail!

$$\begin{aligned}
\int_0^2 \left(x^2 - 2x + 1\right)\, dx &= \int_0^2 x^2\, dx - \int_0^2 2x\, dx + \int_0^2 1\, dx \quad \text{[Linearity.]} \\
&= \left.\frac{x^3}{3}\right|_0^2 - 2\left.\frac{x^2}{2}\right|_0^2 + \left.x\right|_0^2 \quad \text{[Power Rule.]} \\
&= \left(\frac{2^3}{3} - \frac{0^3}{3}\right) - \left(\frac{2^2}{2} - \frac{0^2}{2}\right) + (2-0) \quad \text{[Putting in the numbers.]} \\
&= \frac{8}{3} - 2 + 2 = \frac{8}{3} \quad \text{[Arithmetic.]} \quad \blacksquare
\end{aligned}$$

Quiz #8 *Wednesday, 9 June, 2010. [10 minutes]*

1. Find the area between $y = x\cos\left(x^2\right)$ and the x-axis for $-\sqrt{\frac{\pi}{2}} \le x \le \sqrt{\frac{\pi}{2}}$. *[10]*

SOLUTION. Note that the x-axis is the line $y = 0$. Observe that when $-\sqrt{\frac{\pi}{2}} \le x \le 0$, $0 \le x^2 \le \frac{\pi}{2}$, so $\cos\left(x^2\right) \ge 0$ and hence $x\cos\left(x^2\right) \le 0$. Similarly, when $0 \le x \le \sqrt{\frac{\pi}{2}}$, $0 \le x^2 \le \frac{\pi}{2}$, so $\cos\left(x^2\right) \ge 0$ and hence $x\cos\left(x^2\right) \ge 0$. The area we want, therefore, is the sum of two definite integrals, which we evaluate with the help of the Substitution Rule:

$$\text{Area} = \int_{-\sqrt{\pi/2}}^{0} \left[0 - x\cos\left(x^2\right)\right]\, dx + \int_0^{\sqrt{\pi/2}} \left[x\cos\left(x^2\right) - 0\right]\, dx$$

Using $u = x^2$, so $du = 2x\, dx$, and thus $x\, dx = \frac{1}{2}\, du$, and

changing limits, $\begin{array}{cccc} x & -\sqrt{\pi/2} & 0 & \sqrt{\pi/2} \\ u & \pi/2 & 0 & \pi/2 \end{array}$ we get:

$$= \int_{\pi/2}^{0} -\frac{1}{2}\cos(u)\, du + \int_0^{\pi/2} \frac{1}{2}\cos(u)\, du$$

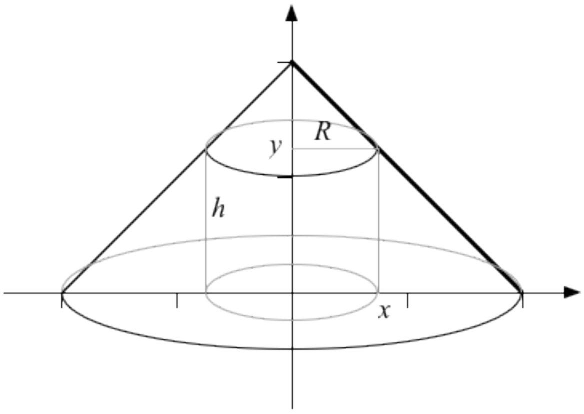
Since $\displaystyle\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$, we now get:

$$\begin{aligned}
&= \int_0^{\pi/2} \frac{1}{2}\cos(u)\, du + \int_0^{\pi/2} \frac{1}{2}\cos(u)\, du \\
&= \int_0^{\pi/2} \cos(u)\, du = \left.\sin(u)\right|_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = 1 \quad \blacksquare
\end{aligned}$$

Quiz #9 *Monday, 14 June, 2010. [10 minutes]*

The region between $y = 2 - x$ and the x-axis, for $0 \le x \le 2$, is rotated about the y-axis. Find the volume of the resulting solid of revolution using both

1. the disk method *[5]* and
2. the method of cylindrical shells. *[5]*



Solution to 1. Note that if x is between 0 and 2, $y = 2 - x$ is also between 0 and 2. The disk at height $y = 2 - x$ would have radius $R = x = 2 - y$. Thus the volume of the solid of revolution in this case is:

$$\begin{aligned}
\int_0^2 \pi R^2\,dy &= \int_0^2 \pi x^2\,dy = \int_0^2 (2-y)^2\,dy \\
&= \int_0^2 \left(4 - 4y + y^2\right)\,dy = \pi\left(4y - 4\frac{y^2}{2} + \frac{y^3}{3}\right)\Big|_0^2 \\
&= \pi\left(4\cdot 2 - 4\frac{4}{2} + \frac{8}{3}\right) - \pi\left(4\cdot 0 - 4\frac{0}{2} + \frac{0}{3}\right) \\
&= \pi\cdot\frac{8}{3} - \pi\cdot 0 = \frac{8}{3}\pi \qquad \blacksquare
\end{aligned}$$

Solution to 2. The cylindrical shell at x would have radius $R = x$ and height $h = y = 2 - x$. Thus the volume of the solid of revolution in this case is:

$$\begin{aligned}
\int_0^2 2\pi Rh\,dx &= \int_0^2 2\pi xy\,dx = \int_0^2 2\pi x(2-x)\,dx \\
&= \int_0^2 2\pi\left(2x - x^2\right)\,dx = 2\pi\left(2\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^2 \\
&= 2\pi\left(2\frac{4}{2} - \frac{8}{3}\right) - 2\pi\left(2\frac{0}{2} - \frac{0}{3}\right) \\
&= 2\pi\cdot\frac{4}{3} - 2\pi\cdot 0 = \frac{8}{3}\pi \qquad \blacksquare
\end{aligned}$$

Quiz #10 *Wednesday, 16 June, 2010. [10 minutes]*

1. Compute $\displaystyle\int_1^e (\ln(x))^2\, dx$. *[10]*

SOLUTION. We will use integration by parts with $u = (\ln(x))^2$ and $v' = 1$, so $u' = 2\ln(x)\cdot\frac{1}{x}$ and $v = x$. Then:

$$\begin{aligned}
\int_1^e (\ln(x))^2\, dx &= \int_1^e u\cdot v'\, dx = u\cdot v|_1^e - \int_1^e v\cdot u'\, dx \\
&= (\ln(x))^2\cdot x\Big|_1^e - \int_1^e x\cdot 2\ln(x)\cdot\frac{1}{x}\, dx \\
&= \left[(\ln(e))^2\cdot e - (\ln(1))^2\cdot 1\right] - 2\int_1^e \ln(x)\, dx
\end{aligned}$$

To solve the remaining integral we use parts again, with $s = \ln(x)$ and $t' = 1$, so $s' = \dfrac{1}{x}$ and $t = x$.

$$\begin{aligned}
&= \left[1^2\cdot e - 0^2\cdot 1\right] - 2\left[\ln(x)\cdot x|_1^e - \int_1^e x\cdot\frac{1}{x}\, dx\right] \\
&= e - 2\left[(\ln(e)\cdot e - \ln(1)\cdot 1) - \int_1^e 1\, dx\right] \\
&= e - 2\left[(1\cdot e - 0\cdot 1) - x|_1^e\right] \\
&= e - 2\left[e - (e-1)\right] \\
&= e - 2 \qquad \blacksquare
\end{aligned}$$

Quiz #11 *Monday, 21 June, 2010. [12 minutes]*

Compute each of the following integrals:

1. $\displaystyle\int_0^{\pi/2} \cos^3(x)\sin^2(x)\, dx$ *[5]* 2. $\displaystyle\int \sec^3(x)\, dx$ *[5]*

SOLUTION TO 1.

$$\begin{aligned}
\int_0^{\pi/2} \cos^3(x)\sin^2(x)\, dx &= \int_0^{\pi/2} \cos^2(x)\sin^2(x)\cos(x)\, dx \\
&= \int_0^{\pi/2} \left(1-\sin^2(x)\right)\sin^2(x)\cos(x)\, dx \\
&= \int_0^{\pi/2} \left(\sin^2(x)-\sin^4(x)\right)\cos(x)\, dx = \int_0^1 \left(u^2-u^4\right)\, du
\end{aligned}$$

Using the substitution $u = \sin(x)$, so $du = \cos(x)\, dx$, and $\begin{matrix} x & 0 & \pi/2 \\ u & 0 & 1 \end{matrix}$.

$$\begin{aligned}
&= \left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right)\Big|_0^1 = \left(\frac{1}{3}1^3 - \frac{1}{5}1^5\right) - \left(\frac{1}{3}0^3 - \frac{1}{5}0^5\right) \\
&= \frac{1}{3} - \frac{1}{5} = \frac{2}{15} \qquad \blacksquare
\end{aligned}$$

SOLUTION I TO 2. We'll use the reduction formula for $\int \sec^n(x)\,dx$,

$$\begin{aligned}\int \sec^3(x)\,dx &= \frac{1}{3-1}\sec^{3-2}(x)\tan(x) + \frac{3-2}{3-1}\int \sec^{3-2}(x)\,dx\\ &= \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\,dx\\ &= \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln\left(\sec(x)+\tan x\right) + C\,,\end{aligned}$$

as well as having memorized a certain notoriously nasty anti-derivative. ■

SOLUTION II TO 2. We'll use integration by parts, with

$$\begin{array}{ll} u = \sec(x) & v' = \sec^2(x)\\ u' = \sec(x)\tan(x) & v = \tan(x)\end{array}$$

rather than apply the reduction formula, and also do a certain notoriously nasty anti-derivative from scratch.

$$\begin{aligned}\int \sec^3(x)\,dx &= \sec(x)\tan(x) - \int \tan(x)\sec(x)\tan(x)\,dx\\ &= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx\\ &= \sec(x)\tan(x) - \int \sec(x)\left(\sec^2(x)-1\right)\,dx\\ &= \sec(x)\tan(x) - \int \left(\sec^3(x)-\sec(x)\right)\,dx\\ &= \sec(x)\tan(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx\end{aligned}$$

Moving all copies of $\int \sec^3(x)\,dx$ to the left in this equation gives

$$2\int \sec^3(x)\,dx = \sec(x)\tan(x) + \int \sec(x)\,dx\,,$$

so

$$\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\,dx\,.$$

We still need to compute $\int \sec(x)\,dx$:

$$\int \sec(x)\,dx = \int \sec(x)\cdot\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx = \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx$$

We now use the substitution $u = \sec(x)+\tan(x)$, so $du = \left(\sec(x)\tan(x)+\sec^2(x)\right)\,dx$.

$$= \int \frac{1}{u}\,du = \ln(u) + C = \ln\left(\sec(x)+\tan(x)\right) + C$$

Hence

$$\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln\left(\sec(x)+\tan(x)\right) + C\,.$$

Whew! ■

Quiz #12 *Wednesday, 23 June, 2010. [15 minutes]*

Compute each of the following integrals:

1. $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$ *[5]* 2. $\displaystyle\int_1^2 x\sqrt{x^2-1}\,dx$ *[5]*

SOLUTION TO 1. Since we see see an expression of the form $\frac{1}{\sqrt{4-x^2}}$, we will use the trig substitution $x = 2\sin(\theta)$, so $dx = 2\cos(\theta)\,d\theta$ and $\theta = \arcsin\left(\frac{x}{2}\right)$.

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{\sqrt{4-4\sin^2(\theta)}}2\cos(\theta)\,d\theta = \int \frac{2\cos(\theta)}{\sqrt{4\cos^2(\theta)}}\,d\theta$$
$$= \int \frac{2\cos(\theta)}{2\cos(\theta)}\,d\theta = \int 1\,d\theta = \theta + C = \arcsin\left(\frac{x}{2}\right) + C \qquad \blacksquare$$

SOLUTION I TO 2. *[The hard way.]* Since we see an expression of the form $\sqrt{x^2-1}$, we will use the trig substitution $x = \sec(\theta)$, so $dx = \sec(\theta)\tan(\theta)\,d\theta$ and $\begin{array}{ccc} x & 1 & 2 \\ \theta & 0 & \frac{\pi}{3} \end{array}$. (Recall that $\cos(0) = 1$, so $\sec(0) = 1$, and $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, so $\sec\left(\frac{\pi}{3}\right) = 2$.)

$$\int_1^2 x\sqrt{x^2-1}\,dx = \int_0^{\pi/3} \sec(\theta)\sqrt{\sec^2(\theta)-1}\,\sec(\theta)\tan(\theta)\,d\theta$$
$$= \int_0^{\pi/3} \sec(\theta)\sqrt{\tan^2(\theta)}\,\sec(\theta)\tan(\theta)\,d\theta$$
$$= \int_0^{\pi/3} \sec(\theta)\tan(\theta)\sec(\theta)\tan(\theta)\,d\theta$$
$$= \int_0^{\pi/3} \tan^2(\theta)\sec^2(\theta)\,d\theta = \int_0^{\sqrt{3}} u^2\,du$$

Using the substitution $u = \tan(\theta)$, so $du = \sec^2(\theta)\,d\theta$ and $\begin{array}{ccc} \theta & 0 & \pi/3 \\ u & 0 & \sqrt{3} \end{array}$.

$$= \frac{1}{3}u^3\Big|_0^{\sqrt{3}} = \frac{1}{3}\left(\sqrt{3}\right)^3 - \frac{1}{3}0^3 = \frac{1}{3}3\sqrt{3} - 0 = \sqrt{3} \qquad \blacksquare$$

SOLUTION II TO 2. *[The easier way.]* Note that the derivative of $w = x^2 - 1$ is $\frac{dw}{dx} = 2x$ and that we have an x outside the square root. We will therefore use the substitution $w = x^2 - 1$, so $dw = 2x\,dx$ and $\begin{array}{ccc} x & 1 & 2 \\ w & 0 & 3 \end{array}$. Note that $\frac{1}{2}dw = x\,dx$.

$$\int_1^2 x\sqrt{x^2-1}\,dx = \int_0^3 \sqrt{w}\,\frac{1}{2}dw = \frac{1}{2}\int_0^3 w^{1/2}\,dw = \frac{1}{2}\cdot\frac{w^{3/2}}{3/2}\Big|_0^3$$
$$= \frac{1}{3}w^{3/2}\Big|_0^3 = \frac{1}{3}\left(\sqrt{3}\right)^3 - \frac{1}{3}0^3 = \frac{1}{3}3\sqrt{3} - 0 = \sqrt{3} \qquad \blacksquare$$

Quiz #13 *Monday, 28 June, 2010. [12 minutes]*

1. Compute $\displaystyle\int \frac{2x^2+3}{(x^2+4)(x-1)}\,dx$

SOLUTION. This is a job for partial fractions. Note that the polynomial in the numerator has lower degree than the polynomial in the denominator, and that the latter is already factored into an irreducible quadratic (as $x^2+4>0$ for all x) and a linear term.

First, we rewrite the rational function using partial fractions:

$$\begin{aligned}\frac{2x^2+3}{(x^2+4)(x-1)} &= \frac{Ax+B}{x^2+4}+\frac{C}{x-1}\\ &= \frac{(Ax+B)(x-1)+C\left(x^2+4\right)}{(x^2+4)(x-1)}\\ &= \frac{Ax^2-Ax+Bx-B+Cx^2+4C}{(x^2+4)(x-1)}\\ &= \frac{(A+C)x^2+(B-A)x+(4C-B)}{(x^2+4)(x-1)}\end{aligned}$$

Comparing the coefficients in the numerators at the beginning and the end gives us a system of linear equations,

$$\begin{array}{rrrrrrrr} & A & & & + & C & = & 2\\ - & A & + & B & & & = & 0\\ & & - & B & + & 4C & = & 3\,,\end{array}$$

which we solve. From the second equation, we know that $A=B$. Substituting this in gives us a system of two equations:

$$\begin{array}{rrrrrr} & A & + & C & = & 2\\ - & A & + & 4C & = & 3\end{array}$$

If we add these two, A disappears and we are left with $5C=5$, so $C=1$. Substituting this into $A+C=2$ gives us $A=1$, and since $A=B$, it follows also that $B=1$. Thus

$$\frac{2x^2+3}{(x^2+4)(x-1)} = \frac{x+1}{x^2+4}+\frac{1}{x-1}\,.$$

Second, we split the integral up accordingly,

$$\begin{aligned}\frac{2x^2+3}{(x^2+4)(x-1)}\,dx &= \int\frac{x+1}{x^2+4}\,dx+\int\frac{1}{x-1}\,dx\\ &= \int\frac{x}{x^2+4}\,dx+\int\frac{1}{x^2+4}\,dx+\int\frac{1}{x-1}\,dx\,,\end{aligned}$$

and work on the pieces.

For $\int \frac{x}{x^2+4}\,dx$ we use the substitution $u = x^2 + 4$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. Then

$$\int \frac{x}{x^2+4}\,dx = \int \frac{1}{u} \cdot \frac{1}{2}\,du = \frac{1}{2}\ln(u) + K = \frac{1}{2}\ln\left(x^2+4\right) + K\,,$$

where K is the generic constant.

For $\int \frac{1}{x^2+4}\,dx$ we use the trig substitution $x = 2\tan(\theta)$, so $dx = 2\sec^2(\theta)\,d\theta$ and $\theta = \arctan\left(\frac{x}{2}\right)$. Then, using the identity $1 + \tan^2(\theta) = \sec^2(\theta)$ at the key step,

$$\begin{aligned}\int \frac{1}{x^2+4}\,dx &= \int \frac{1}{4\tan^2(\theta)+4} \cdot 2\sec^2(\theta)\,d\theta = \frac{2}{4}\int \frac{\sec^2(\theta)}{\sec^2(\theta)}\,d\theta \\ &= \frac{1}{2}\int 1\,d\theta = \frac{1}{2}\theta + L = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + L\,,\end{aligned}$$

where L is the generic constant.

For $\int \frac{1}{x-1}\,dx$ we use the subsitution $w = x - 1$, so $dw = dx$. Then

$$\int \frac{1}{x-1}\,dx = \int \frac{1}{w}\,dw = \ln(w) + M = \ln(x-1) + M\,,$$

where M is the generic constant.

It follows that

$$\begin{aligned}\int \frac{2x^2+3}{(x^2+4)(x-1)}\,dx &= \int \frac{x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx + \int \frac{1}{x-1}\,dx \\ &= \frac{1}{2}\ln\left(x^2+4\right) + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + \ln(x-1) + N\,,\end{aligned}$$

where $N = K + L + M$ is the combined generic constant. ■

Quiz #14 *Wednesday, 30 June, 2010. [10 minutes]*

1. Compute $\displaystyle\int_0^\infty \frac{1}{x^2+1}\,dx$. *[10]*

SOLUTION. This is obviously an improper integral, so we need to take a limit:

$$\int_0^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{x^2+1}\,dx$$

Using the trig substitution $x = \tan(\theta)$, so that $dx = \sec^2(\theta)\,d\theta$, and keeping the limits for x gives:

$$= \lim_{t\to\infty}\int_{x=0}^{x=t} \frac{1}{\tan^2(\theta)+1}\sec^2(\theta)\,d\theta$$

Using the identity $\tan^2(\theta)+1 = \sec^2(\theta)$ gives:

$$= \lim_{t\to\infty}\int_{x=0}^{x=t} \frac{1}{\sec^2(\theta)}\sec^2(\theta)\,d\theta$$

$$= \lim_{t\to\infty}\int_{x=0}^{x=t} 1\,d\theta = \lim_{t\to\infty} \theta\Big|_{x=0}^{x=t}$$

Substituting back:

$$= \lim_{t\to\infty} \arctan(x)\Big|_{x=0}^{x=t}$$

$$= \lim_{t\to\infty} \left(\arctan(t) - \arctan(0)\right)$$

Since $\tan(0) = 0$ we also have $\arctan(0) = 0$.

$$= \lim_{t\to\infty} \arctan(t) = \frac{\pi}{2}$$

Since $\tan(\theta)$ has a vertical asymptote at $\frac{\pi}{2}$. ■

Quiz #15 *Monday, 5 July, 2010. [10 minutes]*

1. Compute the arc-length of the curve $y = \frac{2}{3}x^{3/2}$, where $0 \le x \le 1$.

SOLUTION. First, $\dfrac{dy}{dx} = \dfrac{2}{3}\cdot\dfrac{3}{2}x^{1/2} = x^{1/2}$. Plugging this into the arc-length formula gives:

$$\text{Arc-length} = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1+\left(x^{1/2}\right)^2}\,dx = \int_0^1 \sqrt{1+x}\,dx$$

Substitute $u = x+1$, so $du = dx$ and $\begin{array}{ccc} x & 0 & 1 \\ u & 1 & 2 \end{array}$.

$$= \int_1^2 \sqrt{u}\,du = \int_1^2 u^{1/2}\,du = \left.\frac{u^{3/2}}{3/2}\right|_1^2 = \frac{2}{3}2^{3/2} - \frac{2}{3}1^{3/2}$$

$$= \frac{2}{3}2\sqrt{2} - \frac{2}{3}1 = \frac{2}{3}\left(2\sqrt{2}-1\right) \qquad ■$$

Quiz #16 *Wednesday, 7 July, 2010. [15 minutes]*

1. Find the arc-length of the parametric curve $x = t\cos(t)$ and $y = t\sin(t)$, where $0 \le t \le 1$. *[10]*

SOLUTION. First, using the Chain Rule, we compute

$$\begin{aligned} \frac{dx}{dt} &= 1 \cdot \cos(t) + t \cdot (-\sin(t)) = \cos(t) - t\sin(t) \quad \text{and} \\ \frac{dy}{dt} &= 1 \cdot \sin(t) + t \cdot \sin(t) = \sin(t) + t\cos(t)\,. \end{aligned}$$

Second, we plug this into the arc-length formula for parametric curves:

$$\begin{aligned} \text{Arc-length} &= \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^1 \sqrt{(\cos(t) - t\sin(t))^2 + (\sin(t) + t\cos(t))^2}\, dt \\ &= \int_0^1 \sqrt{\begin{array}{l}\cos^2(t) - 2t\cos(t)\sin(t) + t^2\sin^2(t) \\ \quad + \sin^2(t) + 2t\cos(t)\sin(t) + t^2\cos^2(t)\end{array}}\, dt \\ &= \int_0^1 \sqrt{\left(\cos^2(t) + \sin^2(t)\right)(1 + t^2)}\, dt = \int_0^1 \sqrt{1+t^2}\, dt \\ &\qquad \text{Substitute } t = \tan(\theta) \text{, so } dt = \sec^2(\theta)\, d\theta \\ &\qquad \text{and } \begin{array}{ccc} t & 0 & 1 \\ \theta & 0 & \pi/4 \end{array}. \\ &= \int_0^{\pi/4} \sqrt{1 + \tan^2(\theta)}\, \sec^2(\theta)\, d\theta \\ &= \int_0^{\pi/4} \sqrt{\sec^2(\theta)}\, \sec^2(\theta)\, d\theta = \int_0^{\pi/4} \sec^3(\theta)\, d\theta \\ &\qquad \text{This can be done by parts, or looking it up,} \\ &\qquad \text{or even doing Quiz \#11 over again. :-)} \\ &= \left[\frac{1}{2}\tan(\theta)\sec(\theta) - \frac{1}{2}\ln\left(\tan(\theta) + \sec(\theta)\right)\right]\Bigg|_0^{\pi/4} \\ &\qquad \text{Recall that } \tan(\pi/4) = 1 \text{ and } \sec(\pi/4) = \sqrt{2}, \\ &\qquad \text{while } \tan(0) = 0 \text{ and } \sec(0) = 1 \text{, and } \ln(1) = 0. \\ &= \left[\frac{1}{2}1\sqrt{2} - \frac{1}{2}\ln\left(1 + \sqrt{2}\right)\right] - \left[\frac{1}{2}0 \cdot 1 - \frac{1}{2}\ln(0+1)\right] \\ &= \frac{1}{2}\sqrt{2} - \frac{1}{2}\ln\left(1 + \sqrt{2}\right) \qquad \blacksquare \end{aligned}$$

Quiz #17 *Monday, 12 July, 2010. [15 minutes]*

1. Sketch the curve given by $r = \sin(\theta)$, $0 \le \theta \le \pi$, in polar coordinates. *[2]*
2. Sketch the curve given by $r = \sin(\theta)$, $\pi \le \theta \le 2\pi$, in polar coordinates. *[2]*
3. Find the area of the region enclosed by the curve given by $r = \sin(\theta)$, $0 \le \theta \le \pi$, in polar coordinates. *[6]*

Bonus: Find an equation in Cartesian coordinates for the curve given by $r = \sin(\theta)$, $0 \le \theta \le \pi$, in polar coordinates. *[2]*

SOLUTION TO 1. Note that as θ changes from 0 to π, r increases from 0 to 1 (at $\theta = \pi/2$), and then decreases to 0 again. See Figure 1 below. ■

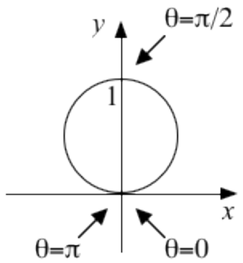


Figure 1

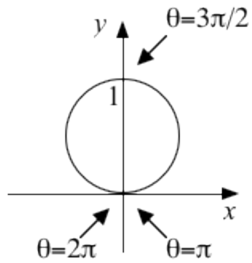


Figure 2

SOLUTION TO 2. Note that as θ changes from π to 2π, r decreases from 0 to -1 (at $\theta = \pi/2$), and then increases to 0 again. Recall that a negative r is interpreted as being $|r|$ units from the origin in the opposite direction. See Figure 2 above. ■

SOLUTION TO 3. We plug $r = \sin(\theta)$ into the area formula for polar coordinates and chug away:

$$\begin{aligned}\int_0^\pi \frac{1}{2} r^2 \, d\theta &= \frac{1}{2}\int_0^\pi \sin^2(\theta)\, d\theta = \frac{1}{2}\int_0^\pi \frac{1-\cos(2\theta)}{2}\, d\theta = \frac{1}{4}\int_0^\pi (1-\cos(2\theta))\, d\theta \\ &= \frac{1}{4}\left(\theta - \frac{1}{2}\sin(2\theta)\right)\Big|_0^\pi = \frac{1}{4}\left(\pi - \frac{1}{2}\sin(2\pi)\right) - \frac{1}{4}\left(0 - \frac{1}{2}\sin(2\cdot 0)\right) \\ &= \frac{1}{4}(\pi - 0) - \frac{1}{4}(0-0) = \frac{\pi}{4} \qquad ■\end{aligned}$$

SOLUTION TO THE BONUS. Observe that for this curve $x = r\cos(\theta) = \sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta)$ and $y = r\sin(\theta) = \sin(\theta)\sin(\theta) = \sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$, so

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\sin(2\theta)\right)^2 + \left(\frac{1}{2} - \frac{1}{2}\cos(2\theta) - \frac{1}{2}\right)^2 = \frac{1}{4}\sin^2(2\theta) + \frac{1}{4}\cos^2(2\theta) = \frac{1}{4}.$$

That is, the curve is a circle of radius $\frac{1}{2}$ and centre $\left(0, \frac{1}{2}\right)$, which is compatible with the sketch in Figure 1 above. ■

Quiz #18 *Wednesday, 14 July, 2010. [12 minutes]*

1. Use the definition of convergence of a series to compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. *[10]*

   Hint: Note that $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$.

SOLUTION. By definition, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to a number L if the partial sums $S_n = \displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)}$ have a limit of L as $n \to \infty$. We check to see what happens when we take the limit of the partial sums:

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i(i+1)} \\
&= \lim_{n\to\infty} \left(\frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \frac{1}{3(3+1)} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)} \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)} \right) \\
&\qquad \text{(Now the hint comes in at last!)} \\
&= \lim_{n\to\infty} \left(\left[\frac{1}{1} - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] + \cdots + \left[\frac{1}{n-1} - \frac{1}{n}\right] + \left[\frac{1}{n} - \frac{1}{n+1}\right] \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{1} + \left[-\frac{1}{2} + \frac{1}{2}\right] + \left[-\frac{1}{3} + \cdots + \frac{1}{n-1}\right] + \left[-\frac{1}{n} + \frac{1}{n}\right] - \frac{1}{n+1} \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{1} - \frac{1}{n+1} \right) = 1 - 0 = 1 \qquad \text{since } \frac{1}{n+1} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

It follows that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and $= 1$. ■

Quiz #19 *Monday, 19 July, 2010. [10 minutes]*

1. Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converges or diverges. *[10]*

SOLUTION. Observe that $\dfrac{1}{n^2+1}$ is a ratio of polynomials in n with the degree of the denominator being $2 = 2 - 0$ more than the degree of the numerator. Since $2 > 1$, it follows by the Generalized p-Test that the series converges. ■

Note: This series can also be shown to converge by using the Integral Test or the Comparison Test (or one of its several variants and extensions).

Quiz #20 *Wednesday, 14 July, 2010. [12 minutes]*

1. Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{1+n}$ converges conditionally, converges absolutely, or diverges. *[10]*

SOLUTION. This series converges conditionally. To see that it does converge, we apply the Alternating Series Test.

Zeroth, observe that $\cos(0) = 1$, $\cos(\pi) = -1$, $\cos(2\pi) = 1$, $\cos(3\pi) = -1$, and so on. In general, $\cos(n\pi) = (-1)^n$.

First, note that the series survives the Divergence Test:

$$\lim_{n\to\infty} \left| \frac{\cos(n\pi)}{1+n} \right| = \lim_{n\to\infty} \frac{|(-1)^n|}{1+n} = \lim_{n\to\infty} \frac{1}{1+n} = 0\,,$$

since $1 + n \to \infty$ as $n \to \infty$.

Second, since $n + 2 > n + 1$ for $n \geq 0$,

$$\left| \frac{\cos(n\pi)}{1+n} \right| = \frac{1}{n+1} > \frac{1}{n+2} = \left| \frac{\cos\left((n+1)\pi\right)}{1+n} \right|,$$

so the terms of the series are decreasing in absolute value.

Third, since $\cos(n\pi) = (-1)^n$ and since $\frac{1}{1+n} > 0$ when $n \geq 0$, it follows that $\frac{\cos(n\pi)}{1+n}$ alternates sign as n increases, *i.e.* $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{1+n}$ is an alternating series.

It follows that $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{1+n}$ converges by the Alternating Series Test. To see that it does not converge absolutely, consider the corresponding series of absolute values, $\displaystyle\sum_{n=0}^{\infty} \left| \frac{\cos(n\pi)}{1+n} \right| = \sum_{n=0}^{\infty} \frac{1}{1+n}$. Since $\dfrac{1}{1+n}$ is a ratio of polynomials in n with the degree of the denominator being $1 = 1 - 0$ more than the degree of the numerator. Since $1 \leq 1$, it follows by the Generalized p-Test that the series of absolute values diverges.

Thus $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{1+n}$ converges conditionally. ■

Quiz #21 *Monday, 26 July, 2010. [15 minutes]*

1. Find the radius and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{n3^n}{2^{n+1}} x^n$. *[10]*

SOLUTION. We will use the Ratio Test to find the radius of convergence:

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{\frac{(n+1)3^{n+1}}{2^{n+2}} x^{n+1}}{\frac{n3^n}{2^{n+1}} x^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)3^{n+1}2^{n+1}x^{n+1}}{n3^n2^{n+2}x^n}\right| \\
&= \lim_{n\to\infty} \left|\frac{3(n+1)x}{2n}\right| = \frac{3|x|}{2} \lim_{n\to\infty} \frac{n+1}{n} \\
&= \frac{3|x|}{2} \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = \frac{3|x|}{2}(1+0) = \frac{3|x|}{2}
\end{aligned}$$

It follows by the Ratio Test that $\displaystyle\sum_{n=0}^{\infty} \frac{n3^n}{2^{n+1}} x^n$ converges absolutely when $\dfrac{3|x|}{2} < 1$, *i.e.* when $|x| < \dfrac{2}{3}$, and diverges when $\dfrac{3|x|}{2} > 1$, *i.e.* when $|x| > \dfrac{2}{3}$. Thus the radius of convergence of the series is $R = \dfrac{2}{3}$.

As $R < \infty$, we need to determine whether the series converges at $x = \pm R = \pm\frac{2}{3}$ to find the interval of convergence. That is, we need to determine whether the series

$$\sum_{n=0}^{\infty} \frac{n3^n}{2^{n+1}} \left(-\frac{2}{3}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{n}{2} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{n3^n}{2^{n+1}} \left(\frac{2}{3}\right)^n = \sum_{n=0}^{\infty} \frac{n}{2}$$

converge or diverge. Since

$$\left.\begin{array}{l} \displaystyle\lim_{n\to\infty} \left|(-1)^n \frac{n}{2}\right| \\ \displaystyle\lim_{n\to\infty} \frac{n}{2} \end{array}\right\} = \frac{1}{2} \lim_{n\to\infty} n = \infty \neq 0\,,$$

both of these series diverge by the Divergence Test. Thus the interval of convergence of the given series is $\left(-\frac{2}{3}, \frac{2}{3}\right)$. ∎

Quiz #22 *Wednesday, 28 July, 2010. [15 minutes]*

1. Find the Taylor series of $f(x) = \ln(x)$ at $a = 1$. *[10]*

SOLUTION I. *(Using Taylor's formula.)* We take the successive derivatives of $f(x) = \ln(x)$ and evaluate them at $a = 1$:

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln(x)$	0
1	x^{-1}	1
2	$-x^{-2}$	-1
3	$2x^{-3}$	2
4	$-6x^{-4}$	-6
5	$24x^{-5}$	24
$\vdots$	$\vdots$	$\vdots$

In general, when $n \geq 1$, we have $f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}$ and so $f^{(n)}(1) = (-1)^{n-1}(n-1)!$. Note that $f^{(0)}(1) = \ln(1) = 0$.

It follows that the Taylor series of $f(x) = \ln(x)$ at $a = 1$ is

$$\sum_{n=1}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots . \quad \blacksquare$$

SOLUTION II. *(Using underhanded cunning.)* Observe that $\frac{d}{dx}\ln(x) = \frac{1}{x} = \frac{1}{1-(1-x)}$. Using the formula for the sum of a geometric series, we get:

$$\frac{1}{x} = \frac{1}{1-(1-x)} = 1 + (1-x) + (1-x)^2 + (1-x)^3 + \cdots$$

$$= 1 - (x-1) + (x-1)^2 - (x-3)^3 + \cdots = \sum_{k=0}^{\infty} (-1)^k (x-1)^k$$

By the uniqueness of Taylor series, this must be the Taylor series at $a = 1$ of $\frac{1}{x} = \frac{d}{dx}\ln(x)$. Integrating this series term-by-term gives the Taylor series at $a = 0$ of $\ln(x)$, at least up to a constant:

$$\sum_{k=0}^{\infty} \int (-1)^k (x-1)^k \, dx = C + \sum_{k=0}^{\infty} \frac{(-1)^k (x-1)^{k+1}}{k+1}$$

$$= C + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} - \cdots$$

Since this series should equal $\ln(1) = 0$ when $x = 1$, we must have $C = 0$.

Thus the Taylor series of $f(x) = \ln(x)$ at $a = 1$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$. (Here we've changed indices, with $n = k + 1$, to make it look like the previous solution.) $\blacksquare$