Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2010
Solutions to the Final Examination

Part I. Do all three (3) of **1**–**3**.

1. Compute $\frac{dy}{dx}$ as best you can in any *three* (3) of **a**–**f**. *[15 = 3 × 5 each]*

a. $x^2 + 3xy + y^2 = 23$ **b.** $y = \ln(\tan(x))$ **c.** $y = \int_x^3 \ln(\tan(t))\, dt$

d. $y = \frac{e^x}{e^x - e^{-x}}$ **e.** $\begin{array}{l} x = \cos(2t) \\ y = \sin(3t) \end{array}$ **f.** $y = (x+2)e^x$

SOLUTIONS TO **1**. Using various tricks!

a. Implicit differentiation and some algebra:

$$\begin{aligned} x^2 + 3xy + y^2 = 23 \quad &\Longrightarrow \quad \frac{d}{dx}\left(x^2 + 3xy + y^2\right) = \frac{d}{dx}23 \\ &\Longrightarrow \quad 2x + 3y + 3x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \\ &\Longrightarrow \quad (2x+3y) + (3x+2y)\frac{dy}{dx} = 0 \\ &\Longrightarrow \quad \frac{dy}{dx} = -\frac{2x+3y}{3x+2y} \quad \blacksquare \end{aligned}$$

b. Chain Rule:

$$\frac{dy}{dx} = \frac{d}{dx}\ln(\tan(x)) = \frac{1}{\tan(x)} \cdot \frac{d}{dx}\tan(x) = \cot(x)\sec^2(x) \quad \blacksquare$$

c. The Fundamental Theorem of Calculus:

$$\frac{dy}{dx} = \frac{d}{dx}\int_x^3 \ln(\tan(t))\, dt = \frac{d}{dx}(-1)\int_3^x \ln(\tan(t))\, dt = -\ln(\tan(x)) \quad \blacksquare$$

d. Quotient Rule and some algebra with e^x:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{e^x}{e^x - e^{-x}}\right) = \frac{\left(\frac{d}{dx}e^x\right)\left(e^x - e^{-x}\right) - e^x \frac{d}{dx}\left(e^x - e^{-x}\right)}{\left(e^x - e^{-x}\right)^2} \\ &= \frac{e^x\left(e^x - e^{-x}\right) - e^x\left(e^x + e^{-x}\right)}{\left(e^x - e^{-x}\right)^2} = \frac{e^{2x} - e^0 - e^{2x} - e^0}{\left(e^x - e^{-x}\right)^2} \\ &= \frac{-2}{\left(e^x - e^{-x}\right)^2} \qquad \text{Note that } \frac{d}{dx}e^{-x} = -e^{-x}. \ \blacksquare \end{aligned}$$

e. As usual with parametric functions:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\cos(2t)}{\frac{d}{dt}\sin(3t)} = \frac{\sin(2t)\cdot(-2)}{\cos(3t)\cdot 3} = -\frac{2\sin(2t)}{3\cos(3t)}$$

… and there's not much one can try to do to simplify this that doesn't make it worse. ■

f. Product Rule:

$$\frac{dy}{dx} = \frac{d}{dx}\left((x+2)e^x\right) = \left(\frac{d}{dx}(x+2)\right)\cdot e^x + (x+2)\cdot\frac{d}{dx}e^x = 1e^x + (x+2)e^x = (x+3)e^x \quad \blacksquare$$

2. Evaluate any *three* (3) of the integrals **a**–**f**. *[15 = 3 × 5 each]*

$$\textbf{a.}\ \int_{-\pi/4}^{\pi/4} \tan(x)\,dx \quad \textbf{b.}\ \int \frac{1}{t^2-1}\,dt \quad \textbf{c.}\ \int_0^{\pi} x\cos(x)\,dx$$

$$\textbf{d.}\ \int \sqrt{w^2+9}\,dw \quad \textbf{e.}\ \int_1^e \ln(x)\,dx \quad \textbf{f.}\ \int \frac{e^x}{e^{2x}+2e^x+1}\,dx$$

SOLUTIONS TO **2**. Using various tricks!

a. We'll write $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and take it from there.

$$\int_{-\pi/4}^{\pi/4} \tan(x)\,dx = \int_{-\pi/4}^{\pi/4} \frac{\sin(x)}{\cos(x)}\,dx$$

Substitute $u = \cos(x)$, so $du = -\sin(x)\,dx$ and $(-1)\,du = \sin(x)\,dx$. Also, $\begin{array}{ccc} x & -\pi/4 & \pi/4 \\ u & 1/\sqrt{2} & 1/\sqrt{2} \end{array}$.

$$= \int_{1/\sqrt{2}}^{1/\sqrt{2}} \frac{-1}{u}\,du = 0 \quad \blacksquare$$

b. This one can be done with the trig substitution $t = \tan(\theta)$, but that approach requires integrating $\csc(\theta)$ along the way. We will use partial fractions instead. Note first that $t^2 - 1 = (t-1)(t+1)$. Then

$$\frac{1}{t^2-1} = \frac{A}{t-1} + \frac{B}{t+1},$$

which requires that $1 = A(t+1) + B(t-1) = (A+B)t + (A-B)$, *i.e.* $A + B = 0$ and $A - B = 1$. Adding the last two equations gives $2A = 1$, so $A = \frac{1}{2}$, and substituting back into either equation and solving for B gives $B = -\frac{1}{2}$. Hence

$$\begin{aligned}\int \frac{1}{t^2-1}\,dt &= \int \left(\frac{1/2}{t-1} - \frac{1/2}{t+1}\right)dt \\ &= \frac{1}{2}\int \frac{1}{t-1}\,dt - \frac{1}{2}\int \frac{1}{t+1}\,dt \\ &= \frac{1}{2}\ln(t-1) - \frac{1}{2}\ln(t+1) + C = \frac{1}{2}\ln\left(\frac{t-1}{t+1}\right) + C. \quad \blacksquare\end{aligned}$$

c. This is a job for integration by parts. We'll use $u = x$ and $v' = \cos(x)$, so $u' = 1$ and $v = \sin(x)$. Thus

$$\begin{aligned}
\int_0^\pi x\cos(x)\,dx &= \int_0^\pi uv'\,dx = uv|_0^\pi - \int_0^\pi u'v\,dx \\
&= x\sin(x)|_0^\pi - \int_0^\pi \sin(x)\,dx \\
&= \pi\sin(\pi) - 0\sin(0) - (-\cos(x))|_0^\pi \\
&= \pi\cdot 0 - 0\cdot 0 + \cos(\pi) - \cos(0) = 0 - 0 - 1 - 1 = -2\,. \quad \blacksquare
\end{aligned}$$

d. This is a job for a trig substitution, namely $w = 3\tan(\theta)$, so $dw = 3\sec^2(\theta)\,d\theta$.

$$\begin{aligned}
\int \sqrt{w^2+9}\,dw &= \int \sqrt{9\tan^2(\theta)+9}\cdot 3\sec^2(\theta)\,d\theta \\
&= \int 3\sqrt{\tan^2(\theta)+1}\cdot 3\sec^2(\theta)\,d\theta \\
&= 9\int \sqrt{\sec^2(\theta)}\cdot\sec^2(\theta)\,d\theta = 9\int \sec^3(\theta)\,d\theta \\
&\quad\text{This last we look up rather than do it from scratch \dots} \\
&= \frac{9}{2}\sec(\theta)\tan(\theta) + \frac{9}{2}\ln\left(\sec(\theta)+\tan(\theta)\right) + C \\
&= \qquad \text{Substituting back, } \tan(\theta) = \frac{w}{3} \text{ and } \sec(\theta) = \sqrt{1+\frac{w^2}{9}}. \\
&= \frac{9}{2}\cdot\frac{w}{3}\sqrt{1+\frac{w^2}{9}} + \frac{9}{2}\ln\left(\frac{w}{3}+\sqrt{1+\frac{w^2}{9}}\right) + C \\
&= \frac{3w}{2}\sqrt{1+\frac{w^2}{9}} + \frac{9}{2}\ln\left(\frac{w}{3}+\sqrt{1+\frac{w^2}{9}}\right) + C \quad \blacksquare
\end{aligned}$$

e. Integration by parts again, with $u = \ln(x)$ and $v' = 1$, so $u' = \dfrac{1}{x}$ and $v = x$.

$$\begin{aligned}
\int_1^e \ln(x)\,dx &= \int_1^e uv'\,dx = uv|_1^e - \int_1^e u'v\,dx \\
&= x\ln(x)|_1^e - \int_1^e \frac{1}{x}\cdot x\,dx \\
&= e\ln(e) - 1\ln(1) - \int_1^e 1\,dx \\
&= e\cdot 1 - 1\cdot 0 - x|_1^e \\
&= e - (e-1) = e - e + 1 = 1 \quad \blacksquare
\end{aligned}$$

f. Substitute $u = e^x$, so $du = e^x\,dx$ and $e^{2x} = (e^x)^2 = u^2$, and see what happens:

$$\int \frac{e^x}{e^{2x} + 2e^x + 1}\,dx = \int \frac{1}{u^2 + 2u + 1}\,du = \int \frac{1}{(u+1)^2}\,du$$

Substitute again, with $w = u + 1$ and $dw = du$.

$$= \int \frac{1}{w^2}\,dw = \frac{-2}{w^3} + C$$

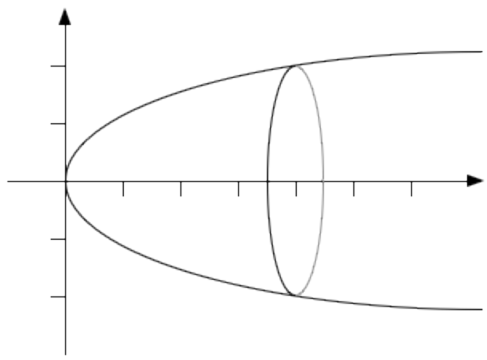
Now we undo the substitutions.

$$= -\frac{2}{(u+1)^3} + C = -\frac{2}{(e^x+1)^3} + C \quad \blacksquare$$

3. Do any *five* (5) of **a**–**i**. *[25 = 5 × 5 ea.]*

a. Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $0 \le x \le 4$, the x-axis, and $x = 4$, about the x-axis.

SOLUTION. Here's a crude sketch of the solid:



We'll use the disk/washer method. The disk at x has radius $R = \sqrt{x} - 0 = \sqrt{x}$; since it is a disk rather than a washer, we need not worry about an inner radius. The the volume of the solid is

$$\int_0^4 \pi R^2\,dx = \int_0^4 \pi\left(\sqrt{x}\right)^2\,dx = \pi\int_0^4 x\,dx$$
$$= \pi\frac{x^2}{2}\Big|_0^4 = \pi\frac{4^2}{2} - \pi\frac{0^2}{2} = 8\pi - 0\pi = 8\pi\,,$$

cube units of whatever sort. $\blacksquare$

b. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x\to 1} 3x = 3$.

SOLUTION. We need to show that for any $\varepsilon > 0$ there is a $\delta > 0$ such that, for all x, if $|x - 1| < \delta$, then $|3x - 3| < \varepsilon$.

Given an $\varepsilon > 0$, we obtain the required $\delta > 0$ with some reverse-engineering:

$$|3x - 3| < \varepsilon \quad \Longleftrightarrow \quad |3(x-1)| < \varepsilon \quad \Longleftrightarrow \quad |x - 1| < \frac{\varepsilon}{3}$$

Since each step is reversible, it follows that if we let $\delta = \frac{\varepsilon}{3}$, then $|3x - 3| < \varepsilon$.

c. Find the Taylor series of $f(x) = \dfrac{x^2}{1 - x^2}$ at $a = 0$ without taking any derivatives.

Solution. Recall that the formula for the sum of the geometric series $\sum_{n=0}^{\infty} ar^n$ with first term s and common ratio $r < 1$ is $\dfrac{s}{1-r}$. If we set $s = x^2$ and $r = x^2$, it now follows that

$$\frac{x^2}{1-x^2} = \sum_{n=0}^{\infty} x^2 \left(x^2\right)^n = \sum_{n=0}^{\infty} x^{2n+2} \,,$$

at least when $\left|x^2\right| < 1$, *i.e.* when $|x| < 1$. By the uniqueness of power series representations, it follows that $\sum_{n=0}^{\infty} x^{2n+2}$ is the Taylor series at 0 of $f(x) = \dfrac{x^2}{1-x^2}$. ■

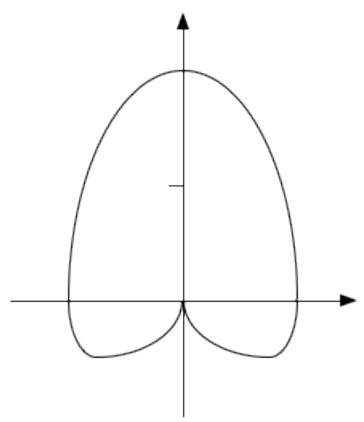
d. Sketch the polar curve $r = 1 + \sin(\theta)$ for $0 \le \theta \le 2\pi$.

Solution. The simplest way to do this is to compute some points on the curve and connect up the dots.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\sin(\theta)$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
r	1	$3/2$	$1{+}1/\sqrt{2}$	$1{+}\sqrt{3}/2$	2	$1{+}\sqrt{3}/2$	$1{+}1/\sqrt{2}$	$3/2$	1

$7\pi/6$	$5\pi/4$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$7\pi/4$	$11\pi/6$	2π
$-1/2$	$-1/\sqrt{2}$	$-\sqrt{3}/2$	-1	$-\sqrt{3}/2$	$-1/\sqrt{2}$	$-1/2$	0
$1/2$	$1{-}1/\sqrt{2}$	$1{-}\sqrt{3}/2$	0	$1{-}\sqrt{3}/2$	$1{-}1/\sqrt{2}$	$1/2$	1

Here's a rough sketch of the curve:



■

e. Use the limit definition of the derivative to compute $f'(1)$ for $f(x) = x^2$.

SOLUTION. Here goes:

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h} \\ &= \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h} = \lim_{h \to 0} \frac{2h + h^2}{h} \\ &= \lim_{h \to 0} (2 + h) = 2 \quad \blacksquare \end{aligned}$$

f. Use the Right-hand Rule to compute the definite integral $\displaystyle\int_1^2 \frac{x}{2}\,dx$.

SOLUTION. We plug into the Right-hand Rule formula and chug away:

$$\begin{aligned} \int_1^2 \frac{x}{2}\,dx &= \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2-1}{n} \cdot \frac{1 + i\frac{2-1}{n}}{2} = \lim_{n \to \infty} \frac{1}{2n} \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right) \\ &= \lim_{n \to \infty} \frac{1}{2n} \left[\left(\sum_{i=1}^{n} 1 \right) + \left(\frac{1}{n} \sum_{i=1}^{n} i \right) \right] = \lim_{n \to \infty} \frac{1}{2n} \left[n + \frac{1}{n} \cdot \frac{n(n+1)}{2} \right] \\ &= \lim_{n \to \infty} \left[\frac{1}{2n} \cdot n + \frac{1}{2n} \cdot \frac{n+1}{2} \right] = \lim_{n \to \infty} \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{4n} \right] \\ &= \frac{3}{4} + 0 = \frac{3}{4} \quad \blacksquare \end{aligned}$$

g. Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges absolutely, converges conditionally, or diverges.

SOLUTION. The series converges by the Alternating Series Test: First,

$$\lim_{n \to \infty} |a_n| = \lim_{n \to \infty} \left| \frac{(-1)^n}{\ln(n)} \right| = \lim_{n \to \infty} \frac{1}{\ln(n)} = 0\,,$$

since $\ln(n) \to \infty$ as $n \to \infty$. Second, since $\ln(n)$ is an increasing function of n, we have that

$$|a_{n+1}| = \left| \frac{(-1)^{n+1}}{\ln(n+1)} \right| = \frac{1}{\ln(n+1)} < \frac{1}{\ln(n)} = \left| \frac{(-1)^n}{\ln(n)} \right| = |a_n|\,.$$

Third, since $\ln(n) > 0$ when $n \geq 2$ and $(-1)^n$ alternates sign, this is an alternating series.

On the other hand, the Comparison Test shows the series does not converge absolutely. Note that $n > \ln(n)$ for $n \geq 2$, so

$$\frac{1}{n} < \frac{1}{\ln(n)} = \left| \frac{(-1)^n}{\ln(n)} \right|\,.$$

Since the harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, it follows that the series $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln(n)} \right|$ diverges as well. Thus the given series does not converge absolutely.

Since it converges, but not absolutely, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges conditionally. ■

h. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{n^2}{\pi^n} x^n$.

SOLUTION. We will use the Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{(n+1)^2}{\pi^{n+1}} x^{n+1}}{\frac{n^2}{\pi^n} x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \cdot \frac{x}{\pi} \right| \\
&= \frac{|x|}{\pi} \lim_{n\to\infty} \frac{n^2+2n+1}{n^2} = \frac{|x|}{\pi} \lim_{n\to\infty} \left(1 + \frac{2}{n} + \frac{1}{n^2} \right) \\
&= \frac{|x|}{\pi} (1+0+0) = \frac{|x|}{\pi}
\end{aligned}$$

It follows by the Ratio Test that $\sum_{n=0}^{\infty} \frac{n^2}{\pi^n} x^n$ converges if $\frac{|x|}{\pi} < 1$, *i.e.* if $|x| < \pi$, and diverges if $\frac{|x|}{\pi} > 1$, *i.e.* if $|x| > \pi$, so the radius of convergence of the series is $R = \pi$. ■

i. Compute the arc-length of the polar curve $r = \theta$, $0 \le \theta \le 1$.

SOLUTION. We plug the given curve into the polar version of the arc-length formula and chug away. Note that $\frac{dr}{d\theta} = 1$ if $r = \theta$.

$$\text{Length} = \int_0^1 \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^1 \sqrt{\theta^2 + 1^2}\, d\theta = \int_0^1 \sqrt{\theta^2+1}\, d\theta$$

We use the trig substitution $\theta = \tan(t)$,

so $d\theta = \sec^2(t)\,dt$ and $\begin{array}{ccc} \theta & 0 & 1 \\ t & 0 & \pi/4 \end{array}$

$$= \int_0^{\pi/4} \sqrt{\tan^2(t)+1}\, \sec^2(t)\,dt = \int_0^{\pi/4} \sqrt{\sec^2(t)}\, \sec^2(t)\,dt = \int_0^{\pi/4} \sec^3(t)\,dt$$

As in the solution to **2d**, we look this up.

$$\begin{aligned}
&= \frac{1}{2}\sec(t)\tan(t) + \frac{1}{2}\ln\left(\sec(t)+\tan(t)\right)\Big|_0^{\pi/4} \\
&= \frac{1}{2}\sec(\pi/4)\tan(\pi/4) + \frac{1}{2}\ln\left(\sec(\pi/4)+\tan(\pi/4)\right) \\
&\quad - \frac{1}{2}\sec(0)\tan(0) - \frac{1}{2}\ln\left(\sec(0)+\tan(0)\right) \\
&= \frac{1}{2}\cdot\sqrt{2}\cdot 1 + \frac{1}{2}\ln\left(\sqrt{2}+1\right) - \frac{1}{2}\cdot 1\cdot 0 - \frac{1}{2}\ln(1+0) = \frac{1}{\sqrt{2}} + \frac{1}{2}\ln\left(\sqrt{2}+1\right)
\end{aligned}$$ ■

Part II. Do any *two* (2) of **4–6**.

4. Find the domain, all maximum, minimum, and inflection points, and all vertical and horizontal asymptotes of $f(x) = e^{-x^2}$, and sketch its graph. *[15]*

SOLUTION. We'll run through the usual checklist and then graph $f(x) = e^{-x^2}$.

i. Domain. Note that both $g(x) = e^x$ and $h(x) = -x^2$ are defined and continuous for all x. It follows that $f(x) = g(h(x)) = e^{-x^2}$ is also defined and continuous for all x. It follows that the domain of $f(x)$ is all of $\mathbb{R}$ and that it has no vertical asymptotes. $\square$

ii. Intercepts. Since $g(x) = e^x$ is never 0, $f(x) = e^{-x^2}$ can never equal 0 either, so it has no x-intercepts. For the y-intercept, simply note that $f(0) = e^{-0^2} = e^0 = 1$. $\square$

iii. Asymptotes. As noted above, $f(x) = e^{-x^2}$ has no vertical asymptotes, so we only need to check for horizontal asymptotes.

$$\lim_{x\to\infty} e^{-x^2} = \lim_{x\to\infty} \frac{1}{e^{x^2}} = 0 \quad \text{and} \quad \lim_{x\to-\infty} e^{-x^2} = \lim_{x\to-\infty} \frac{1}{e^{x^2}} = 0\,,$$

since $e^{x^2} \to \infty$ as $x^2 \to \infty$, which happens as $x \to \pm\infty$. Thus $f(x) = e^{-x^2}$ has the horizontal asymptote $y = 0$ in both directions. $\square$

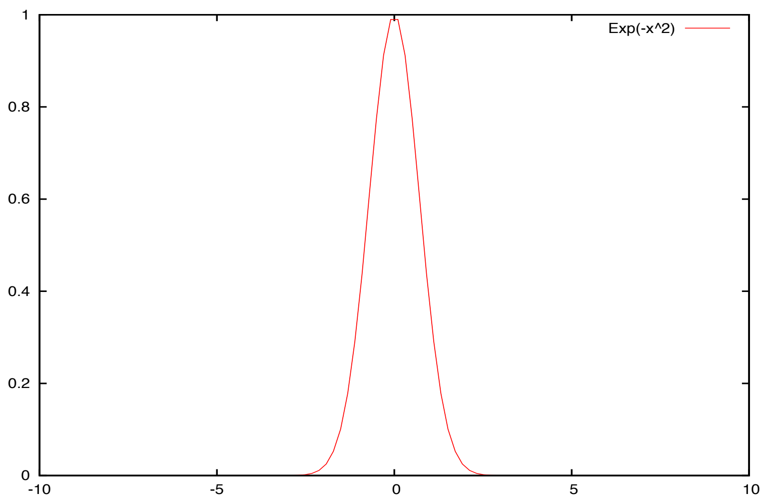
iv. Maxima and minima. $f'(x) = e^{-x^2} \dfrac{d}{dx}\left(-x^2\right) = -2xe^{-x^2}$, which equals 0 exactly when $x = 0$ because $-2e^{-x^2} \neq 0$ for all x. Note that this is the only critical point. Since $e^{-x^2} > 0$ for all x, $f'(x) = -2xe^{-x^2} > 0$ when $x < 0$ and < 0 when $x > 0$, so $f(x) = e^{-x^2}$ is increasing for $x < 0$ and decreasing for $x > 0$. Thus $x = 0$ is an (absolute!) maximum point of $f(x)$, which has no minimum points. $\square$

v. Inflection points.

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(-2xe^{-x^2}\right) = -2e^{-x^2} - 2x\frac{d}{dx}\left(-x^2\right) \\ &= -2e^{-x^2} - 2x\cdot\left(-2xe^{-x^2}\right) = \left(4x^2-2\right)e^{-x^2}\,, \end{aligned}$$

which equals which equals 0 exactly when $4x^2 - 2 = 0$, *i.e.* when $x = \pm\frac{1}{\sqrt{2}}$, because $-2e^{-x^2} \neq 0$ for all x. Since $e^{-x^2} > 0$ for all x, $f''(x) = \left(4x^2-2\right)e^{-x^2} > 0$ exactly when $4x^2 - 2 > 0$, *i.e.* when $|x| > \frac{1}{\sqrt{2}}$, and is < 0 exactly when $4x^2 - 2 < 0$, *i.e.* when $|x| < \frac{1}{\sqrt{2}}$. Thus $f(x) = e^{-x^2}$ is concave up on $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$ and concave down on $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Thus $f(x) = e^{-x^2}$ has two inflection points, at $x = \pm\frac{1}{\sqrt{2}}$. $\square$

v. Graph. $f(x) = e^{-x^2}$ is essentially the classic "bell curve" without some small adjustments that are made to have the total area under the "bell curve" be equal to 1.

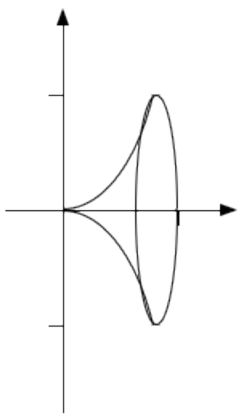


This graph was generated with the command tt Plot2D(Exp(-x^2),-10:10) in `Yacas` ("Yet Another Computer Algebra System"). □

That's all for this one, folks! ■

5. Find the area of the surface obtained by rotating the curve $y = \tan(x)$, $0 \leq x \leq \dfrac{\pi}{4}$, about the x-axis. *[15]*

SOLUTION. This is, quite unintentionally, by far the hardest problem on the exam. [I hallucinated my way to a fairly simple "solution" when making up the exam, and the error survived all my checks ...] Here's a crude sketch of the surface:



Note that $\dfrac{dy}{dx} = \dfrac{d}{dx}\tan(x) = \sec^2(x)$. Plugging this into the appropriate surface area formula gives:

$$\int_0^{\pi/4} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{\pi/4} 2\pi\tan(x)\sqrt{1+\sec^4(x)}\,dx$$

Let $u = \sec^2(x)$, so $du = 2\sec(x)\tan(x)\,dx$ and

$2\tan(x)\,dx = \dfrac{1}{sec^2(x)}\,du = \dfrac{1}{u}\,du$; also $\begin{array}{ccc} x & 0 & \pi/4 \\ u & 1 & 2 \end{array}$.

$$= \pi\int_1^2 \sqrt{1+u^2}\cdot\frac{1}{u}\,du = \pi\int_1^2 \frac{1}{u}\sqrt{1+u^2}\,du$$

Now let $u = \tan(\theta)$, so $du = \sec^2(\theta)\,d\theta$ and

$\begin{array}{ccc} u & 1 & 2 \\ \theta & \pi/4 & \arctan(2) \end{array}$.

$$= \int_{\pi/4}^{\arctan(2)} \frac{1}{\tan(\theta)}\sqrt{1+\tan^2(\theta)}\,\sec^2(\theta)\,d\theta$$

$$= \int_{\pi/4}^{\arctan(2)} \frac{\sec^3(\theta)}{\tan(\theta)}\,d\theta = \int_{\pi/4}^{\arctan(2)} \frac{1}{\sin(\theta)\cos^2(\theta)}\,d\theta$$

At this point – if they even got this far – most people would get stuck. We have one last desperate option, though, namely the Weierstrauss substitution: $t = \tan\left(\dfrac{\theta}{2}\right)$, so $\cos(\theta) = \dfrac{1-t^2}{1+t^2}$, $\sin(\theta) = \dfrac{2t}{1+t^2}$, and $d\theta = \dfrac{2}{1+t^2}\,dt$. The limits get pretty ugly here, though: $\begin{array}{ccc} \theta & \pi/4 & \arctan(2) \\ t & \tan(\pi/8) & \tan\left(\arctan(2)/2\right) \end{array}$. [There may be some way to simplify the limits, but by now I can't be bothered …] Resuming integration:

$$\int_0^{\pi/4} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_{\pi/4}^{\arctan(2)} \frac{1}{\sin(\theta)\cos^2(\theta)}\,d\theta$$

$$= \int_{\tan(\pi/8)}^{\tan(\arctan(2)/2)} \frac{1+t^2}{2t}\cdot\left(\frac{1+t^2}{1-t^2}\right)^2\cdot\frac{2}{1+t^2}\,dt$$

After some algebra, which I'll let you do, we get

$$= \int_{\tan(\pi/8)}^{\tan(\arctan(2)/2)} \frac{t^4+2t^2+1}{t(t-1)^2(t+1)^2}\,dt$$

… which we can do using partial fractions.

To continue we need to find the constants A–E such that

$$
\begin{aligned}
\frac{t^4+2t^2+1}{t(t-1)^2(t+1)^2} &= \frac{A}{t}+\frac{B}{(t-1)^2}+\frac{C}{t-1}+\frac{D}{(t+1)^2}+\frac{E}{t+1} \\
&= \frac{\begin{array}{r} A(t-1)^2(t+1)^2+Bt(t+1)^2 \\ +\,Ct(t-1)(t+1)^2+Dt(t-1)^2 \\ +\,Et(t-1)^2(t+1) \end{array}}{t(t-1)^2(t+1)^2} \\
&= \frac{\begin{array}{r} (A+C+E)t^4+(B+C+D-E)t^3 \\ +\,(-2A+2B-C-2D-E)t^2 \\ +\,(B-C+D+E)t+A \end{array}}{t(t-1)^2(t+1)^2}\,,
\end{aligned}
$$

that is, satisfying the system of linear equations:

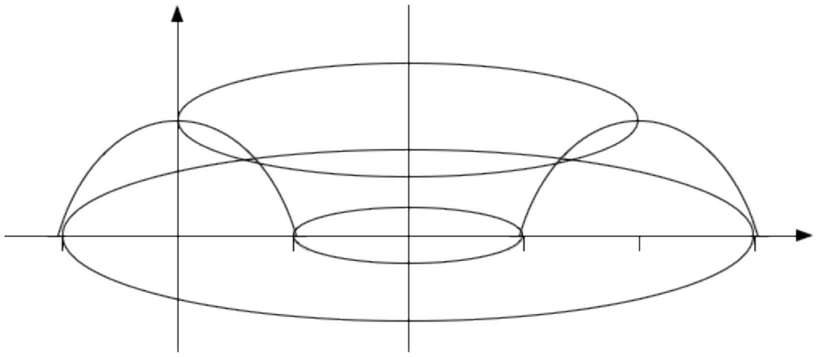
$$
\begin{array}{rrrrrrrrrrrr}
 & A & & & + & C & & & + & E & = & 1 \\
 & & & B & + & C & + & D & - & E & = & 0 \\
- & 2A & + & 2B & - & C & - & 2D & - & E & = & 2 \\
 & & & B & - & C & + & D & + & E & = & 0 \\
 & A & & & & & & & & & = & 1
\end{array}
$$

Solving this [more work for you!] gives us: $A=1$, $B=1$, $C=0$, $D=-1$, and $E=0$. Resuming integration again [and leaving some more routine work for you]:

$$
\begin{aligned}
\int_0^{\pi/4} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx &= \int_{\tan(\pi/8)}^{\tan(\arctan(2)/2)} \frac{t^4+2t^2+1}{t(t-1)^2(t+1)^2}\,dt \\
&= \int_{\tan(\pi/8)}^{\tan(\arctan(2)/2)} \left(\frac{1}{t}+\frac{1}{(t-1)^2}-\frac{1}{(t+1)^2}\right)dt \\
&= \left(\ln(t)-\frac{1}{t-1}+\frac{1}{t+1}\right)\Bigg|_{\tan(\pi/8)}^{\tan(\arctan(2)/2)} \\
&= \Bigg(\ln\left(\tan\left(\arctan(2)/2\right)\right)-\frac{1}{\tan\left(\arctan(2)/2\right)-1} \\
&\qquad +\frac{1}{\tan\left(\arctan(2)/2\right)+1}\Bigg) \\
&\quad -\left(\ln\left(\tan(\pi/8)\right)-\frac{1}{\tan(\pi/8)-1}+\frac{1}{\tan(\pi/8)+1}\right)
\end{aligned}
$$

Simplify if you can — and dare! ■

6. Find the volume of the solid obtained by rotating the region below $y=1-x^2$, $-1\le x\le 1$, and above the x-axis about the line $x=2$. *[15]*



SOLUTION. Here's a crude sketch of the solid:

We will use the method of cylindrical shells to find the volume of this solid. Note that the shell at x, where $-1 \leq x \leq 1$, has radius $r = 2 - x$ and height $h = y - 0 = 1 - x^2$. Plugging this into the formula for the volume gives:

$$\begin{aligned}
\int_{-1}^{1} 2\pi r h \, dx &= \int_{-1}^{1} 2\pi (2-x)\left(1-x^2\right) \, dx \\
&= 2\pi \int_{-1}^{1} \left(2 - x - 2x^2 + x^3\right) \, dx \\
&= 2\pi \left(2x - \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4\right)\Big|_{-1}^{1} \\
&= 2\pi \left(2 - \frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) - 2\pi \left(-2 - \frac{1}{2} + \frac{2}{3} + \frac{1}{4}\right) \\
&= 2\pi \cdot \frac{13}{12} - 2\pi \cdot \frac{-19}{12} = 2\pi \cdot \frac{32}{12} = \frac{16}{3}\pi \quad \blacksquare
\end{aligned}$$

Part III. Do *one* (1) of **7** or **8**.

7. Do all three (3) of **a**–**c**.

a. Use Taylor's formula to find the Taylor series of e^x centred at $a = -1$. *[7]*

SOLUTION. If $f(x) = e^x$, then $f'(x) = e^x$, $f''(x) - e^x$, and so on; it is pretty easy to see that $f^{(n)}(x) = e^x$ for all $n \geq 0$. It follows that $f^{(n)}(-1) = e^{-1} = \dfrac{1}{e}$ for all $n \geq 0$. Hence the Taylor series of e^x centred at $a = -1$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!} \left(x - (-1)\right)^n = \sum_{n=0}^{\infty} \frac{e^{-1}}{n!}(x+1)^n = \sum_{n=0}^{\infty} \frac{1}{n!e}(x+1)^n \quad \blacksquare$$

b. Determine the radius and interval of convergence of this Taylor series. *[4]*

SOLUTION. We'll use the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!e}(x+1)^{n+1}}{\frac{1}{n!e}(x+1)^n}\right| = \lim_{n\to\infty}\left|\frac{1}{n+1}(x+1)\right|$$
$$= |x+1|\lim_{n\to\infty}\frac{1}{n+1} = |x+1|\cdot 0 = 0$$

It follows that the series converges for any x whatsoever, *i.e.* it has radius of convergence $R=\infty$ and hence has interval of convergence $(-\infty,\infty)$. ■

c. Find the Taylor series of e^x centred at $a=-1$ using the fact that the Taylor series of e^x centred at 0 is $\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\cdots$. *[4]*

SOLUTION. We plug $x-(-1)=x+1$ in for x in e^x and in its Taylor series:

$$e^{x+1} = \sum_{n=0}^{\infty}\frac{(x+1)^n}{n!}$$

Since $e^{x+1}=e^xe$, it follows that

$$e^x = \frac{1}{e}\sum_{n=0}^{\infty}\frac{(x+1)^n}{n!} = \sum_{n=0}^{\infty}\frac{(x+1)^n}{n!e}.$$

Since Taylor series are unique this must be the Taylor series of e^x centred at $a=-1$. ■

8. Do all three (3) of **a**–**c**. You may assume that the Taylor series of $f(x)=\ln(1+x)$ centred at $a=0$ is $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\cdots$.

a. Find the radius and interval of convergence of this Taylor series. *[6]*

SOLUTION. We'll use the Ratio Test to find the radius of convergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}}{n+1}x^{n+1}}{\frac{(-1)^{n+1}}{n}x^n}\right| = \lim_{n\to\infty}\left|-\frac{n}{n+1}x\right|$$
$$= |x|\lim_{n\to\infty}\frac{n}{n+1} = |x|\lim_{n\to\infty}\frac{n/n}{(n+1)/n}$$
$$= |x|\lim_{n\to\infty}\frac{1}{1+1/n} = |x|\cdot\frac{1}{1+0} = |x|\cdot 1 = |x|$$

It follows by the Ratio Test that the given Taylor series converges absolutely when $|x|<1$ and diverges when $|x|>1$, so the radius of convergence is $R=1$.

To determine the interval of convergence, we need to check what happens at $x = \pm R = \pm 1$. Plugging in $x = -1$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(-1)^n = \sum_{n=1}^{\infty} \frac{-1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$$

(*i.e.* the negative of the harmonic series), which diverges by the p-Test. Plugging in $x = 1$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}1^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

(*i.e.* the alternating harmonic series), which converges by the Alternating Series Test, as we've seen in class. Therefore the interval of convergence of the given Taylor series is $(-1, 1]$. ■

b. Use this series to show that $\ln\left(\frac{3}{2}\right) = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}$. *[3]*

SOLUTION. Since a function is equal to its Taylor series within the latter's radius of convergence and $\left|\frac{3}{2} - 1\right| = \frac{1}{2} < 1$, we must have

$$\ln\left(\frac{3}{2}\right) = \ln\left(1 + \frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n},$$

as desired. ■

c. Find an n such that $T_n\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \cdots + \frac{(-1)^{n+1}}{n2^n}$ is guaranteed to be within $0.01 = \frac{1}{100}$ of $\ln\left(\frac{3}{2}\right)$. *[6]*

SOLUTION. We need to find an n such that

$$\left|\ln\left(\frac{3}{2}\right) - T_n\left(\frac{1}{2}\right)\right| = \left|\sum_{i=n+1}^{\infty} \frac{(-1)^{i+1}}{i2^i}\right| < 0.01\,.$$

One could, with some effort, accomplish this by considering the nth remainder term, $R_n\left(\frac{1}{2}\right)$, of the given Taylor series, but in this case there is a simpler approach available. Note that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}$ is an alternating series. It follows from the proof of the Alternating Series Test that

$$\left|\sum_{i=n+1}^{\infty} \frac{(-1)^{i+1}}{i2^i}\right| < \left|\frac{(-1)^{n+2}}{(n+1)2^{n+1}}\right|,$$

so all we need to do is ensure that $\left|\dfrac{(-1)^{n+2}}{(n+1)2^{n+1}}\right| = \dfrac{1}{(n+1)2^{n+1}} < 0.01 = \dfrac{1}{100}$. A little brute force goes a long way here:

n	1	2	3	4	5	$\cdots$
$\frac{1}{(n+1)2^{n+1}}$	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{64}$	$\frac{1}{160}$	$\frac{1}{768}$	$\cdots$

Thus $n = 4$ does the job. (Note that any larger n would serve too.) ■

[Total = 100]

Part IV - Something different. Bonus!

$\mathbf{e^{i\pi}}$**.** Write a haiku touching on caclulus or mathmatics in general. *[2]*

haiku?

seventeen in three:
five and seven and five of
syllables in lines

I HOPE THAT YOU ENJOYED THE COURSE. ENJOY THE REST OF THE SUMMER!