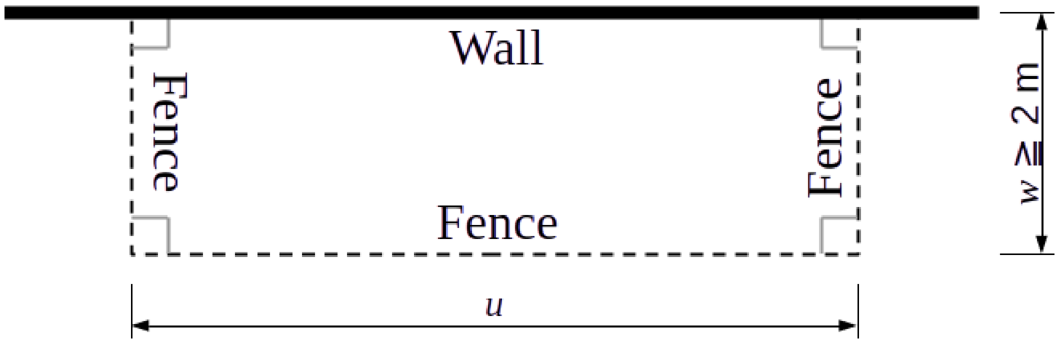
Mathematics 1110H (Section A) – Calculus I: Limits, Derivatives, and Integrals
Trent University, Fall 2024

Solutions to Quiz #6
Max and Min meet Words and Formulas



1. A rectangular enclosure is to be made using a long existing wall as one side of the enclosure and fencing off the other three sides, with the requirement that the side opposite the wall be at least 2 m from the wall. What is the maximum area of such an enclosure if the total amount of fencing available is 60 m? *[3]*

Solution. Suppose u is the length of each of the sides parallel to the wall and w is the length of each of the sides perpendicular to the wall. We need to fence in one side parallel to the wall and both sides perpendicular to the wall; since we have 60 m of fencing, we may assume that $u+2w = 60$, so $u = 60-2w$. (It's obvious, isn't it, that less fencing used means less area enclosed, other things being equal? So we should use all the fencing we have ...) A rectangle with sides of length u and w has area $A = uw = (60-2w)w = 60w-2w^2$. In our case, we are told that we must have $2 \leq w$ and the restriction to 60 m of fencing means that $2w \leq 60$, so $w \leq 30$. Our task, therefore, comes down to maximizing $A(w) = 60w - 2w^2$ for $2 \leq w \leq 30$. Here we go:

Endpoints. $A(2) = 60 \cdot 2-2 \cdot 2^2 = 120-8 = 112$ and $A(30) = 60 \cdot 30-2 \cdot 30^2 = 1800-1800 = 0$.

Critical points. $A'(w) = \frac{d}{dw}\left(60w - 2w^2\right) = 60 - 4w$, which is equal to 0 exactly when $w = \frac{60}{4} = 15$, which is in the interval given by $2 \leq w \leq 30$. At this critical point we have $A(15) = 60 \cdot 15 - 2 \cdot 15^2 = 900 - 2 \cdot 225 = 450$.

Conclusion. Comparing the values at the endpoints of the interval $[2, 30]$ with that at the critical point 15 in this interval, we see that the maximum possible area of a rectangular enclosure meeting the given requirements is 450 m^2. ■

2. Between $0C$ and $30C$ the volume V, in cubic centimetres, of 1 kg of water at temperature T is approximately given by the formula

$$V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3.$$

Find the temperature in the given range at which water has its maximum density. *[2]*

SOLUTION. Since density $= \dfrac{\text{mass}}{\text{volume}}$, a fixed mass of water has maximum density when it has minimum volume. Our task, therefore is to minimize $V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3$ for T with $0 \leq T \leq 30$. Once we have the minimum volume, we can compute the corresponding density. To give credit where much credit is due, all the calculations below were done with the help of a Casio fx -260 Solar II calculator. What it spit out at each stage is what you will see below. It's limitations, though, led to some loss of actual precision in the calculations – don't be fooled too much by the many digits after the decimal points . . .

Endpoints. Here we go:

$$\begin{aligned}
V(0) &= 999.87 - 0.06426 \cdot 0 + 0.0085043 \cdot 0^2 - 0.0000679 \cdot 0^3 \\
&= 999.87 - 0 + 0 - 0 = 999.87 \\
V(30) &= 999.87 - 0.06426 \cdot 30 + 0.0085043 \cdot 30^2 - 0.0000679 \cdot 30^3 \\
&= 999.87 - 0.06426 \cdot 30 + 0.0085043 \cdot 900 - 0.0000679 \cdot 27000 \\
&= 999.87 - 1.9278 + 7.65387 - 1.833 = 1003.76307
\end{aligned}$$

Critical ponts. The derivative of V with respect to T is

$$\begin{aligned}
\frac{dV}{dT} &= \frac{d}{dT}\left(999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3\right) \\
&= 0 - 0.06426 \cdot 1 + 0.0085043 \cdot 2T - 0.0000679 \cdot 3T^2 \\
&= -0.06426 + 0.0170086T - 0.0002037T^2.
\end{aligned}$$

To find out when this expression is equal to 0, we apply the quadratic formula:

$$\begin{aligned}
T &= \frac{-0.0170086 \pm \sqrt{0.0170086^2 - 4 \cdot (-0.0002037) \cdot (-0.06426)}}{2 \cdot (-0.0002037)} \\
&= \frac{-0.0170086 \pm \sqrt{0.000289292474 - 0.000052359048}}{-0.0004074} \\
&= \frac{-0.0170086 \pm \sqrt{0.000236933}}{-0.0004074} = \frac{-0.0170086 \pm 0.015392628}{-0.0004074} \\
&= \frac{0.0170086 \mp 0.015392628}{0.0004074} = \begin{cases} 3.9665488 & \text{if } - \\ 79.53173294 & \text{if } + \end{cases}
\end{aligned}$$

$T = 79.53173294$ is not in the speified interval $[0, 30]$, while $T = 3.9665488$ is, so we need to work out V for $T = 3.9665488$ only:

$$\begin{aligned}
V(3.9665488) &= 999.87 - 0.06426 \cdot 3.9665488 + 0.0085043 \cdot 3.9665488^2 \\
&\qquad - 0.0000679 \cdot 3.9665488^3 \\
&= 999.87 - 0.06426 \cdot 3.9665488 + 0.0085043 \cdot 15.73350938 \\
&\qquad - 0.0000679 \cdot 62.40773276 \\
&= 999.87 - 0.254890425 + 0.133802483 - 0.004237485055 \\
&= 999.74446746
\end{aligned}$$

Conclusion. Comparing the values of V at the endpoints of the interval and at the critical value that is in the interval, we see that V is minimized, and hence density maximized, when $T = 3.9665488$ (well, approximately :-). For this value of T we have density $= \dfrac{\text{mass}}{\text{volume}} = \dfrac{1}{999.74446746} = 0.001000255598\ kg/cm^3$ (approximately, of course). In more sensible units, say g/cm^3, this works out to a density of $1.000255598\ g/cm^3$ (approximately). ■

NOTE. If we were physicists or engineers, we would want to do an error analysis on the above calculation, incorporating also any uncertainties in the expression $V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3$, to understand how reliable the numbers for temperature, volume, and density obtained above actually are. Let's not!