Mathematics 1110H (Section A) – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Fall 2024

Solutions to Quiz #3
Choice!

Please do *one* (1) of questions **1** and **2**. If you do both, only the first one spotted by the grader will be marked.

1. Use the standard version or the game version of the ε–δ definition of limits to verify that $\lim_{x \to 0} (x+1) \neq 2$. *[5]*

Hint: If you don't remember the game version of the ε–δ definition of limits from class, or even if you do, see the handout *An Alternate Version of the ε–δ Definition of Limits*, which you can find in the folder *Textbook and Handouts* in the Course Content section on Blackboard.

SOLUTION. Recall that the game version of the ε–δ definition of limits is as follows:

> The *limit game* for $f(x)$ at $x = a$ with target L is a three-move game played between two players A and B as follows:
>
> 1. A moves first, picking a small number $\varepsilon > 0$.
> 2. B moves second, picking another small number $\delta > 0$.
> 3. A moves third, picking an x that is within δ of a, *i.e.* $a - \delta < x < a + \delta$.
>
> To determine the winner, we evaluate $f(x)$. If it is within ε of the target L, *i.e.* $L - \varepsilon < f(x) < L + \varepsilon$, then player B wins; if not, then player A wins.
>
> With this idea in hand, $\lim_{x \to a} f(x) = L$ means that player B has a winning strategy in the limit game for $f(x)$ at $x = a$ with target L; that is, if B plays it right, B will win no matter what A tries to do. (Within the rules ... :-) Conversely, $\lim_{x \to a} f(x) \neq L$ means that player A is the one with a winning strategy in the limit game for $f(x)$ at $x = a$ with target L.

To show that $\lim_{x \to 0} (x+1) \neq L = 2$, therefore, we need to demonstrate that player A has a winning strategy in the corresponding limit game. Note that the real limit is $\lim_{x \to 0} (x+1) = 0 + 1 = 1$. Here we go:

Move 1. Player A should play an $\varepsilon > 0$ that is small enough to separate the alleged limit of 2 from the actual limit of 1. Since $2 - 1 = 1$, any ε with $0 < \varepsilon < 1$ will do; to keep things as simple as we can, we will have A play $\varepsilon = \frac{1}{2} = 0.5$.

Move 2. The adversary, player B, now plays some $\delta > 0$. We have no control over this ...

Move 3. We will now have player A play $x = 0 - \dfrac{\delta}{2} = -\dfrac{\delta}{2}$.

We claim that player A wins no matter what $\delta > 0$ player B chose. First, observe that because $\delta > 0$, we have $-\delta < -\frac{\delta}{2} - 0 < \delta$, so player A's choice of x is valid within the rules of the limit game. Second, note that $f(x) = f\left(-\frac{\delta}{2}\right) = -\frac{\delta}{2} + 1 < 1 < 2 - \frac{1}{2} = L - \varepsilon$, which means that we do not have $L - \varepsilon < f(x) < L + \varepsilon$, so player B loses, which means that player A wins.

Since player A has a winning strategy – *i.e.* A can win no matter what B does – it follows by the game version of the ε–δ definition of limits that $\lim_{x \to 0} (x+1) \neq 2$. ■

2. Find a single line which is tangent to each of the curves $y = \sin(x)$, $y = \cos(x)$, $y = \sec(x)$, and $y = x^3 + 1$, though not necessarily to all of them at the same point. Explain why the line you give does the job. *[5]*

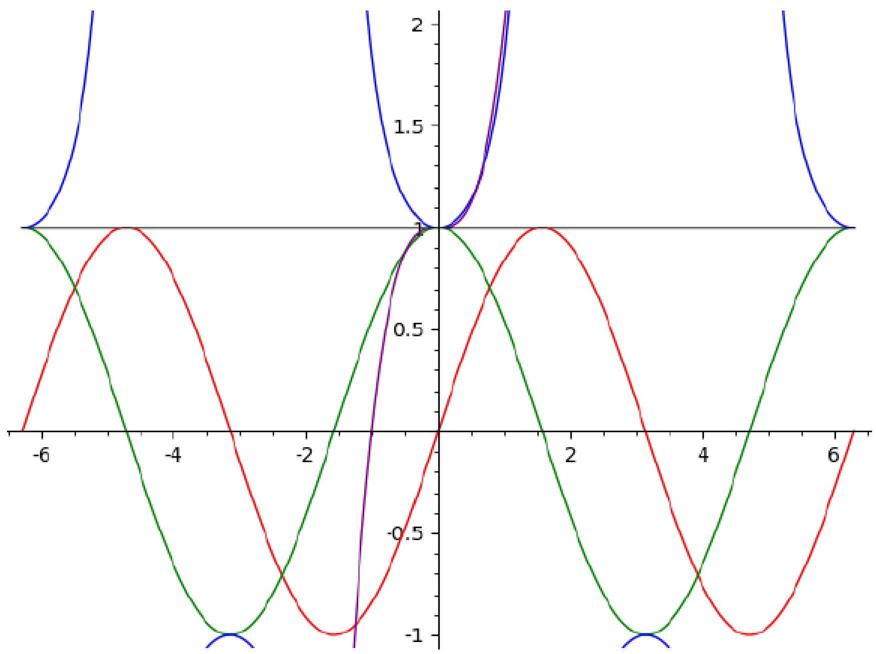
Hint: Draw the graphs of these functions to get an idea of what might work.

SOLUTION. The line $y = 1$ does the job. It touches the graphs of $y = \cos(x)$, $y = \sec(x)$, and $y = x^3 + 1$ at $x = 0$, and the graph of $y = \sin(x)$ at $x = \frac{\pi}{/}2$. The line, being horizontal, has slope 0 (everywhere!) and we leave it to the reader to check that $y = \cos(x)$, $y = \sec(x)$, and $y = x^3 + 1$ all have slope 0 at $x = 0$, and that the graph of $y = \sin(x)$ has slope 0 at $x = \frac{\pi}{2}$.

Following the hint after the fact, here is a graph of all four functions, plus the line $y = 0$. The y-values for $y = \sec(x)$ and $y = x^3 + 1$ have been restricted to $-1 \leq y \leq 2$ to keep the scale under control.

[1]:
```
f1 = plot( sin(x), -2*pi, 2*pi, color='red')
f2 = plot( cos(x), -2*pi, 2*pi, color='green')
f3 = plot( sec(x), -2*pi, 2*pi, color='blue', detect_poles=True, ymin=-1, ymax=2)
f4 = plot( x^3 + 1, -2*pi, 2*pi, color='purple', detect_poles=True, ymin=-1, ymax=2)
f5 = plot( 1, -2*pi, 2*pi, color='black')
f1 + f2 + f3 + f4 + f5
```

[1]:



It should be apparent from this graph that $y = \sin(x)$, $y = \cos(x)$, and $y = \sec(x)$ all have $y = 1$ as a tangent line at (nfinitely many) points other than the ones mentioned in the first paragraph above. ■