Mathematics 1110H (Section A) – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Fall 2024

Solutions to Assignment #1
Epsilonics

1. Verify that $\displaystyle\lim_{x\to 1}\left(-\frac{2x}{3}+\frac{1}{3}\right) = -\frac{1}{3}$ using the ε–δ definition of limits. *[1]*

SOLUTION. To verify that $\displaystyle\lim_{x\to 1}\left(-\frac{2x}{3}+\frac{1}{3}\right) = -\frac{1}{3}$, we need to check that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x-1| < \delta$, then it must be true that $\left|\left(-\frac{2x}{3}+\frac{1}{3}\right)-\left(-\frac{1}{3}\right)\right| < \varepsilon$. As usual, we try to reverse-engineer the necessary δ from the desired outcome. Suppose, then, that $\varepsilon > 0$.

$$\begin{aligned}
\left|\left(-\frac{2x}{3}+\frac{1}{3}\right)-\left(-\frac{1}{3}\right)\right| < \varepsilon &\iff \left|-\frac{2x}{3}+\frac{1}{3}+\frac{1}{3}\right| < \varepsilon \\
&\iff \left|-\frac{2x}{3}+\frac{2}{3}\right| < \varepsilon \\
&\iff \frac{2}{3}\left|-x+1\right| < \varepsilon \\
&\iff \frac{2}{3}\left|x-1\right| < \varepsilon \\
&\iff |x-1| < \frac{\varepsilon}{\frac{2}{3}} \\
&\iff |x-1| < \frac{3}{2}\varepsilon
\end{aligned}$$

This suggests that, given $\varepsilon > 0$, we should take $\delta = \frac{3}{2}\varepsilon$. If $|x-1| < \delta = \frac{3}{2}\varepsilon$, we can run the argument above in reverse – note that every step in it is reversible – to show that it must be true that $\left|\left(-\frac{2x}{3}+\frac{1}{3}\right)-\left(-\frac{1}{3}\right)\right| < \varepsilon$. Thus $\displaystyle\lim_{x\to 1}\left(-\frac{2x}{3}+\frac{1}{3}\right) = -\frac{1}{3}$ by the ε–δ definition of limits. ■

2. Verify that $\displaystyle\lim_{x\to 0} x^2 = 0$ using the ε–δ definition of limits. *[2]*

SOLUTION. We can do this one using a fully-reversible reverse-engineering approach, as in the solution to question **1**, though the algebraic details are different because the function is not linear. To verify that $\displaystyle\lim_{x\to 0} x^2 = 0$, we need to check that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x-0| < \delta$, then it must be true that $\left|x^2-0\right| < \varepsilon$. Suppose, then, that $\varepsilon > 0$.

$$\begin{aligned}
\left|x^2-0\right| < \varepsilon &\iff \left|x^2\right| < \varepsilon \\
&\iff |x|^2 < \varepsilon \\
&\iff |x| < \sqrt{\varepsilon} \\
&\iff |x-0| < \sqrt{\varepsilon}
\end{aligned}$$

This suggests that, given $\varepsilon > 0$, we should take $\delta = \sqrt{\varepsilon}$. If $|x - 0| < \delta = \sqrt{\varepsilon}$, we can run the argument above in reverse, as each step in it is reversible, to show that it must be true that $\left|x^2 - 0\right| < \varepsilon$. Thus $\lim\limits_{x \to 0} x^2 = 0$ by the ε–δ definition of limits. ■

3. Verify that $\lim\limits_{x \to 2} x^2 = 4$ using the ε–δ definition of limits. *[2.5]*

SOLUTION. Here, unfortunately for us, we will have to use an argument that is a bit more subtle than the one in the solution to question **2** because it is not fully reversible. To verify that $\lim\limits_{x \to 2} x^2 = 4$, we need to check that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x - 2| < \delta$, the it must be true that $\left|x^2 - 4\right| < \varepsilon$. Suppose, then, that $\varepsilon > 0$. We'll try to reverse-engineer the necessary δ.

$$\begin{aligned} \left|x^2 - 4\right| < \varepsilon &\iff |(x-2)(x+2)| < \varepsilon \\ &\iff |x-2| \cdot |x+2| < \varepsilon \\ &\iff |x-2| < \frac{\varepsilon}{|x+2|} \end{aligned}$$

At this point we have a problem, in fact, two. First, δ is used to control x, so it can't depend on x, which means that we can't just take $\delta = \frac{\varepsilon}{|x+2|}$. Second, even if we could, we would have to ensure that $|x+2| \neq 0$ – dividing by 0 is undefined – so we can't accidentally allow $x = -2$. We can solve both problems by ensuring up front that δ is small enough to keep x close enough to 2 that it is well away from -2. The distance between 2 and -2 is $2 - (-2) = 4$ and making sure that $\delta < 4$ will do the job. We choose to ensure that $\delta \leq 1$ because 1 is smaller than 4 and easy to work with. What does having $\delta \leq 1$ do for us?

$$\begin{aligned} |x-2| < \delta &\implies |x-2| < 1 \quad \text{(Note that this step is not reversible.)} \\ &\iff -1 < x - 2 < 1 \\ &\iff -1 + 2 < x - 2 + 2 < 1 + 2 \\ &\iff 1 < x < 3 \\ &\iff 3 = 1 + 2 < x + 2 < 3 + 2 = 5 \\ &\iff \frac{1}{3} > \frac{1}{x+2} > \frac{1}{5} \\ &\iff \frac{\varepsilon}{3} > \frac{\varepsilon}{x+2} > \frac{\varepsilon}{5} \end{aligned}$$

Since having $\delta \leq 1$ gives us $x + 2 > 3$, we are guaranteed that $x + 2 > 0$, so $x + 2 \neq 0$ and $|x + 2| = x + 2$. Even better, it also tells us that if $|x - 2| < \delta \leq 1$, then $\frac{\varepsilon}{5} < \frac{\varepsilon}{x+2} = \frac{\varepsilon}{|x+2|}$.

So how exactly do we pick δ? We simply make it be the smaller of 1 and $\frac{\varepsilon}{5}$; that is, $\delta = \min\left\{1, \dfrac{\varepsilon}{5}\right\}$ if you want notation. We check that this works.

Suppose $\varepsilon > 0$ is given and we set $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$. Then $\delta > 0$, as required, since both 1 and $\frac{\varepsilon}{5}$ are greater than 0. Now suppose that $|x - 2| < \delta$. Since $\delta \leq 1$, it follows from the longish calculation above that $\delta \leq \frac{\varepsilon}{5} < \frac{\varepsilon}{|x+2|}$. This, in turn, gives us:

$$\begin{aligned} |x-2| < \delta &\implies |x-2| < \frac{\varepsilon}{5} < \frac{\varepsilon}{|x+2|} \\ &\implies |x-2| < \frac{\varepsilon}{|x+2|} \\ &\iff |x-2| \cdot |x+2| < \varepsilon \\ &\iff |(x-2)(x+2)| < \varepsilon \\ &\iff \left|x^2\right| < \varepsilon \end{aligned}$$

Since for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x-2| < \delta$, then i $\left|x^2 - 4\right| < \varepsilon$, it follows by the ε–δ definition of limits that $\lim_{x \to 2} x^2 = 4$. Whew! ■

4. Consider $\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$.

- **a.** Compute this limit using the practical rules for computing limits. *[1]*
- **b.** Verify that your answer is correct using the ε–δ definition of limits. *[1]*

SOLUTIONS. **a.** Here we go:

$$\lim_{x \to 5} \frac{x^2 - 25}{x - 5} = \lim_{x \to 5} \frac{(x-5)(x+5)}{x-5} = \lim_{x \to 5} (x+5) = 5 + 5 = 10 \quad \square$$

b. Since $\frac{x^2 - 25}{x - 5} = x + 5$ except at $x = 5$ (where it is undefined), this is really a small variation on the procedure in the answer to question **1**. To verify that $\lim_{x \to 5} \frac{x^2 - 25}{x - 5} = 10$ usinfg the ε–δ definition of limits we need to show that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $0 < |x - 5| < \delta$, then $\left| \frac{x^2 - 25}{x - 5} - 10 \right| < \varepsilon$. (We have to ensure that $0 < |x - 5|$ to avoid dividing by 0 in $\frac{x^2 - 25}{x - 5}$.)

As usual, we try to reverse-engineer the required $\delta > 0$. Suppose $\varepsilon > 0$; then:

$$\begin{aligned} \left| \frac{x^2 - 25}{x - 5} - 10 \right| < \varepsilon &\iff \left| \frac{(x-5)(x+5)}{x-5} - 10 \right| < \varepsilon \\ &\iff |x + 5 - 10| < \varepsilon \quad \text{as long as } x \neq 5 \\ &\iff |x - 5| < \varepsilon \end{aligned}$$

It follows that $\delta = \varepsilon$ works. For if $0 < |x - 5| < \delta = \varepsilon$, the above chain of reasoning is fully reversible (as $0 < |x - 5|$ implies that $x \neq 5$), so $\left| \frac{x^2 - 25}{x - 5} - 10 \right| < \varepsilon$. ■

5. Verify that $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$ using the ε–δ definition of limits. *[2.5]*

SOLUTION. The basic idea here is similar to the one in the solution to question **3**: get around an awkward denominator involving x by limiting δ to limit the range of xs we need to deal with.

To verify that $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$ using the ε–δ definition of limits, we need to check that for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $|x - 3| < \delta$, then $\left|\frac{1}{x} - \frac{1}{3}\right| < \varepsilon$. As usual, we try to reverse-engineer the δ we need from $\left|\frac{1}{x} - \frac{1}{3}\right| < \varepsilon$. Suppose, then, that we are given an $\varepsilon > 0$.

$$\begin{aligned}
\left|\frac{1}{x} - \frac{1}{3}\right| < \varepsilon &\iff \left|\frac{3}{3x} - \frac{x}{3x}\right| < \varepsilon \\
&\iff \left|\frac{3-x}{3x}\right| < \varepsilon \\
&\iff \frac{|3-x|}{|3x|} < \varepsilon \\
&\iff \frac{|x-3|}{3|x|} < \varepsilon \\
&\iff |x-3| < 3|x|\varepsilon
\end{aligned}$$

We can't use $\delta = 3|x|\varepsilon$ because it depends on x; there is also a potential problem if $x = 0$, since that would give us that $0 \leq |x - 3| < 0$, and hence that $0 < 0$, which is impossible. We work around this by making sure that δ is small enough to have $|x - 3| < \delta$ imply that $x \neq 0$. The distance between 3 and 0 is 3, so any positive bound on δ that is less than 3 will do. 1 is a convenient positive number less than 3, so we'll use that.

If $0 < \delta \leq 1$ and $|x - 3| < \delta$, then we have $|x - 3| < 1$, so:

$$\begin{aligned}
|x-3| < 1 &\iff -1 < x - 3 < 1 \\
&\iff 2 = -1 + 3 < x < 1 + 3 = 4 \\
&\iff 6 = 3 \cdot 2 < 3x < 2 \cdot 4 = 12 \\
&\iff 6 < 3|x| < 12 \quad (\text{Since } 6 < 3x \Rightarrow 0 < 2 < x \Rightarrow x = |x|.) \\
&\iff 6\varepsilon < 3|x|\varepsilon < 12\varepsilon
\end{aligned}$$

This suggests that we try letting $\delta = \min\{1, 6\varepsilon\}$. Suppose that $|x - 3| < \delta$. Since $\delta \leq 1$, it follows that $\delta \leq 6\varepsilon < 3|x|\varepsilon$. As we then have $|x - 3| < \delta \leq 6\varepsilon < 3|x|\varepsilon$, we can reverse the first long calculation above to obtain $\left|\frac{1}{x} - \frac{1}{3}\right| < \varepsilon$, as required.

Thus $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$ by the ε–δ definition of limits. ∎