Trent University, Fall 2024

MATH 1110H-A Midterm Test Solutions

Wednesday, 30 October

Time: 50 minutes

Name: Less Than Zero

Student Number: -???????

Question	Mark
1	___
2	___
3	___
Total	___ /30

Instructions

- *Show all your work.* Legibly, please! Simplify where you reasonably can.
- *If you have a question, ask it!*
- Use the back sides of all the pages for rough work or extra space.
- You may use a calculator and all sides of one letter- or A4-size aid sheet.
- If you do more than the minimum number of parts or questions, only the first ones the marker finds will be marked. Cross out anything you do not want marked.

1. Do any *two* (2) of parts **a**–**c**. *[10 = 2 × 5 each]*

a. Use the ε–δ definition of limits to check that $\lim\limits_{x \to 1} (2x+1) = 3$.

b. Determine whether $g(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous at $x = 0$.

c. Compute $\lim\limits_{x \to \infty} \dfrac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}}$.

SOLUTIONS. **a.** We need to check that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x-1| < \delta$, then $|(2x+1) - 3| < \varepsilon$. As usual, we start with the desired outcome and try to reverse-engineer the necessary δ. Suppose we are given an $\varepsilon > 0$.

$$\begin{aligned} |(2x+1) - 3| < \varepsilon &\iff |2x - 2| < \varepsilon \\ &\iff |2(x-1)| < \varepsilon \\ &\iff 2\,|x-1| < \varepsilon \\ &\iff |x-1| < \frac{\varepsilon}{2} \end{aligned}$$

It follows that if we set $\delta = \dfrac{\varepsilon}{2}$, then whenever $|x-1| < \delta = \dfrac{\varepsilon}{2}$ we will get $|(2x+1) - 3| < \varepsilon$, since every step in the above process is reversible.

Thus $\lim\limits_{x \to 1} (2x+1) = 3$ by the ε–δ definition of limits. $\square$

b. By the definition of continuity, we need to show that $\lim\limits_{x \to)} g(x) = g(0)$.

Since $-1 \leq \sin(t) \leq 1$ for all t and $-|x| \leq x \leq |x|$ for all x, it follows that

$$-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$$

for all $x \neq 0$. As $\lim\limits_{x \to 0} -|x| = 0 = \lim\limits_{x \to 0} |x|$, it follows by the Squeeze Theorem that

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0 = g(0),$$

so $g(x)$ is continuous at $x = 0$. $\square$

c. We will use a cheap algebraic trick:

$$\begin{aligned} \lim_{x \to \infty} \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} &= \lim_{x \to \infty} \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} \cdot \frac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} \\ &= \lim_{x \to \infty} \frac{\left(\sqrt{x+1}\right)^2 - \left(\sqrt{x-1}\right)^2}{\left(\sqrt{x+1} + \sqrt{x-1}\right)^2} = \lim_{x \to \infty} \frac{(x+1) - (x-1)}{\left(\sqrt{x+1} + \sqrt{x-1}\right)^2} \\ &= \lim_{x \to \infty} \frac{2}{\left(\sqrt{x+1} + \sqrt{x-1}\right)^2} \begin{matrix} \to 2 \\ \to \infty \end{matrix} = 0, \end{aligned}$$

using the fact that $\lim\limits_{x \to \infty} \sqrt{x+1} = \infty = \lim\limits_{x \to \infty} \sqrt{x-1}$. $\blacksquare$

2. Find $\frac{dy}{dx}$ in any *two* (2) of parts **a**–**c**. *[10 = 2 × 5 each]*

a. $y = \cos^2(x) - \sin^2(x)$ **b.** $y = \frac{x^2+1}{x^3}$

c. $y = 2x^2 + 3$ (Using the limit definition of the derivative in this part.)

SOLUTIONS. **a.** *Simplify first.* $y = \cos^2(x) - \sin^2(x) = \cos(2x)$, so, using the Chain Rule,

$$\frac{dy}{dx} = \frac{d}{dx}\cos(2x) = -\sin(2x)\frac{d}{dx}(2x) = -2\sin(2x). \quad \square$$

a. *Just differentiate.* Using the Power and Chain Rules:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\cos^2(x) - \sin^2(x)\right) = 2\cos(x)\frac{d}{dx}\cos(x) - 2\sin(x)\frac{d}{dx}\sin(x) \\ &= 2\cos(x)\left(-\sin(x)\right) - 2\sin(x)\cos(x) = -4\cos(x)\sin(x) \\ &= -2\cdot 2\cos(x)\sin(x) = -2\sin(2x) \quad \square\end{aligned}$$

b. *Simplify first.* $y = \frac{x^2+1}{x^3} = \frac{x^2}{x^3} + \frac{1}{x^3} = \frac{1}{x} + \frac{1}{x^3} = x^{-1} + x^{-3}$, so, using the Power Rule,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{-1} + x^{-3}\right) = (-1)x^{-2} + (-3)x^{-4} = -\frac{1}{x^2} - \frac{3}{x^4} = -\frac{x^2}{x^4} - \frac{3}{x^4} = -\frac{x^2+3}{x^4}. \quad \square$$

b. *Just differentiate.* Using the Quotient and Power Rules:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x^2+1}{x^3}\right) = \frac{\left[\frac{d}{dx}\left(x^2+1\right)\right]x^3 - \left(x^2+1\right)\left[\frac{d}{dx}x^3\right]}{\left(x^3\right)^2} = \frac{2x\cdot x^3 - \left(x^2+1\right)3x^2}{x^6} \\ &= \frac{2x^4 - 3x^4 - 3x^2}{x^6} = -\frac{x^4+3x^2}{x^6} = -\frac{x^2+3}{x^4} \quad \square\end{aligned}$$

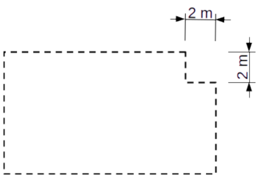
c. We apply the limit definition of the derivative and compute away:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(2x^2+3\right) = \lim_{h\to 0}\frac{\left(2(x+h)^2+3\right) - \left(2x^2+3\right)}{h} \\ &= \lim_{h\to 0}\frac{\left(2\left(x^2+2xh+h^2\right)+3\right) - \left(2x^2+3\right)}{h} = \lim_{h\to 0}\frac{2x^2+4xh+2h^2+3-2x^2-3}{h} \\ &= \lim_{h\to 0}\frac{4xh+2h^2}{h} = \lim_{h\to 0}(4x+2h) = 4x + 2\cdot 0 = 4x \quad \blacksquare\end{aligned}$$

3. Do *one* (1) of parts **a** or **b**. *[10]*

a. Find the domain as well as any – and all! – intercepts, horizontal and vertical asymptotes, intervals of increase and decrease, and local maximum and minimum points, of $f(x) = \dfrac{x^2}{x^2+1}$, and sketch its graph based on this information.

b. An almost rectangular plot is to be fenced off, using exactly 40 m of fencing. It's almost rectangular in that a 2×2 m square is to be left out of one corner of the rectangle, but still fenced, as in the diagram at right. What is the largest possible area of such a plot?



SOLUTIONS. **a.** We run through the indicated checklist:

i. Domain. x^2 and x^2+1 are both defined (and continuous and differentiable) for all x. Since $x^2+1 \geq 0+1 = 1$ for all x, the denominator of the quotient is never 0, so $f(x) = \dfrac{x^2}{x^2+1}$ is defined (and continuous and differentiable) for all x.

ii. Intercepts. Since $f(0) = \dfrac{0^2}{0^2+1} = \dfrac{0}{1} = 0$, $y = f(x)$ has y-intercept 0. As $f(x) = \dfrac{x^2}{x^2+1} = 0$ exactly when $x^2 = 0$, which happens exactly when $x = 0$, $y = f(x)$ has x-intercept 0. Note that the only y-intercept is also the only x-intercept.

iii. Horizontal asymptotes. We take the limits of $f(x)$ as $x \to \pm\infty$ and see what happens.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2}{x^2+1} = \lim_{x\to-\infty} \frac{x^2}{x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{1}{1+\frac{1}{x^2}} \begin{matrix} \to 1 \\ \to 1+0^+ \end{matrix} = \frac{1}{1^+} = 1^-$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2}{x^2+1} = \lim_{x\to+\infty} \frac{x^2}{x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to+\infty} \frac{1}{1+\frac{1}{x^2}} \begin{matrix} \to 1 \\ \to 1+0^+ \end{matrix} = \frac{1}{1^+} = 1^-$$

Thus $y = f(x)$ has $y = 1$ as a horizontal asymptote in both directions, which it approaches from below in both directions.

iv. Vertical asymptotes. Since $f(x) = \dfrac{x^2}{x^2+1}$ is defined and continuous for all x, it cannot have any vertical asymptotes.

v. Intervals of increase and decrease and local maxima and minima. We need to compute the derivative of $f(x)$, which we do using the Quotient and Power Rules:.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{x^2}{x^2+1}\right) = \frac{\left[\frac{d}{dx}x^2\right]\left(x^2+1\right) - x^2\left[\frac{d}{dx}\left(x^2+1\right)\right]}{\left(x^2+1\right)^2} \\ &= \frac{2x\left(x^2+1\right) - x^2 \cdot 2x}{\left(x^2+1\right)^2} = \frac{2x^3+2x-2x^3}{\left(x^2+1\right)^2} = \frac{2x}{\left(x^2+1\right)^2} \end{aligned}$$

Observe that the denominator of $f'(x)$, namely $\left(x^2+1\right)^2$ is defined and positive for all x, so whether $f'(x)$ is positive, zero, or negative is entirely dependent on whether the numerator, namely $2x$, is. That is,

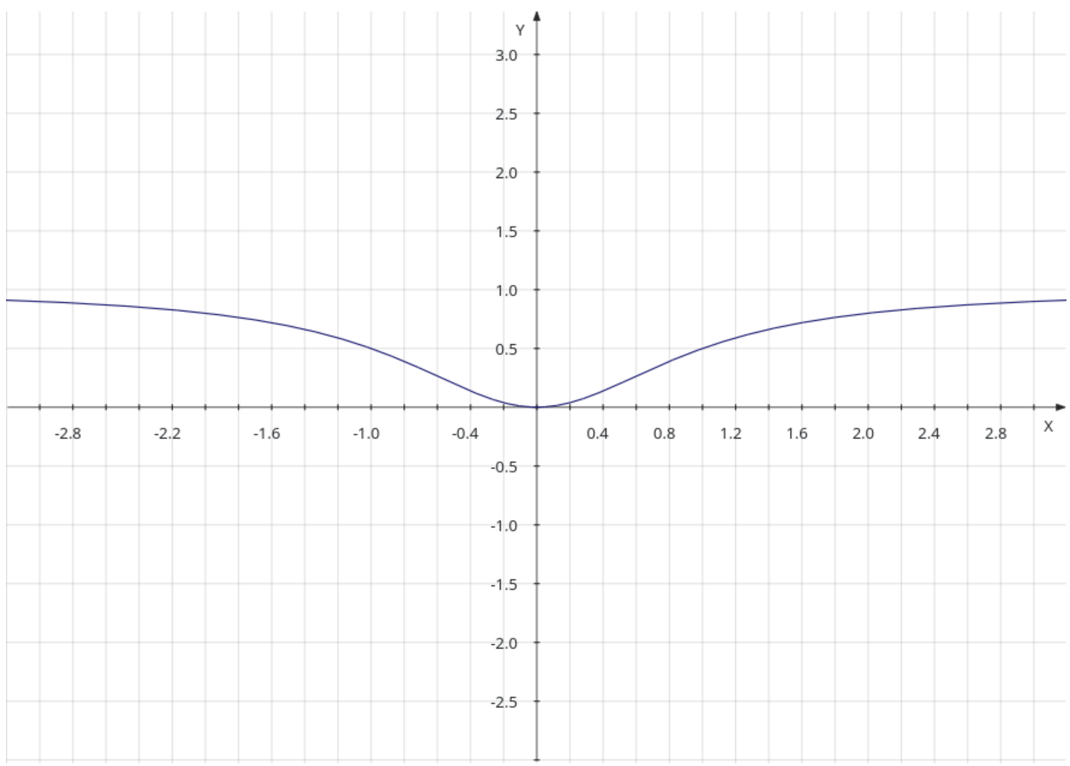
$$f'(x) = \frac{2x}{\left(x^2+1\right)^2} \begin{smallmatrix} <0 \\ =0 \\ >0 \end{smallmatrix} \iff 2x \begin{smallmatrix} <0 \\ =0 \\ >0 \end{smallmatrix} \iff x \begin{smallmatrix} <0 \\ =0 \\ >0 \end{smallmatrix}$$

Thus $f'(x) < 0$, and so $f(x)$ is decreasing, on $(-\infty, 0)$; $f'(x) = 0$, and so $f(x)$ has critical point, at $x = 0$; and $f'(x) > 0$, and so $f(x)$ is increasing, on $(0, \infty)$. Since $f(x)$ is decreasing to the left of the critical point and increasing to the right of the critical point, the critical point at $x = 0$ is a local minimum. (It is also an absolute minimum. Why?) Note that $f(0) = \dfrac{2 \cdot 0}{\left(0^2+1\right)^2} = \dfrac{0}{1} = 0$.

We summarize most of this in the following table:

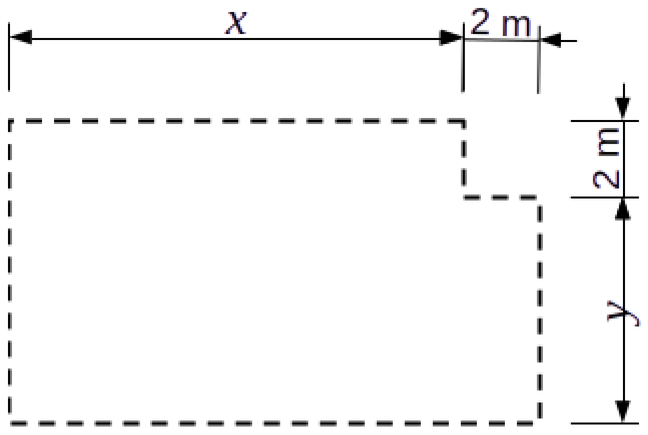
x	$(-\infty, 0)$	0	$(0, \infty)$
$f'(x)$	$-$	0	$+$
$f(x)$	$\downarrow$	min	$\uparrow$

vi. The graph. Cheating ever so slightly, here is the graph, as drawn by a program called `kmplot`. (Can't let SageMath have all the fun!)



■

b. *i. Setup.* Keeping the 2×2 m square cut out of one corner in mind, let $x + 2$ be the length of the rectangular plot and $y + 2$ be its width, as in the diagram below.



The area A of the plot is then the area of an $(x + 2) \times (y + 2)$ rectangle, minus the area of the 2×2 square cut out of one corner, so

$$A = (x + 2)(y + 2) - 2 \cdot 2 = xy + 2x + 2y + 4 - 4 = xy + 2x + 2y.$$

Since the perimeter of a plot of these dimensions is $2(x+2)+2(y+2) = 2x+2y+8$, which should be equal to the 40 m of available fencing, we have

$$2x + 2y + 8 = 40 \implies x + y + 4 = 20 \implies y = -x + 16.$$

This lets us write the area formula for the plot as a function of x alone,

$$\begin{aligned} A(x) &= xy + 2x + 2y = x(-x + 16) + 2x + 2(-x + 16) \\ &= -x^2 + 16x + 2x - 2x + 32 = -x^2 + 16x + 32. \end{aligned}$$

Note that the least x could conceivably be is 0, and the most it could be occurs when $y = 0$, in which case $0 = -x + 16$, so $x = 16$. Thus we must have $0 \leq x \leq 16$.

We can finally get to work on finding the maximum possible area of such a plot; we need to find the maximum of $A(x) = -x^2 + 16x + 32$ for $0 \leq x \leq 16$.

ii. Endpoints. We evaluate $A(x)$ at $x = 0$ and $x = 16$.

$$\begin{aligned} A(0) &= -0^2 + 16 \cdot 0 + 32 = -0 + 0 + 32 = 32 \\ A(16) &= -16^2 + 16 \cdot 16 + 32 = -256 + 256 + 32 = 32 \end{aligned}$$

iii. Critical points. Since $A(x) = -x^2 + 16x + 32$ is defined (and continuous and differentiable) for all x, we need only consider those critical points arising from its having a derivative of 0 in the interval $[0, 16]$. We differentiate $A(x)$ using the Power Rule:

$$A'(x) = \frac{d}{dx}\left(-x^2 + 16x + 32\right) = -2x + 16 + 0 = -2x + 16$$

Then

$$A'(x) = -2x + 16 = 0 \implies x = \frac{-16}{-2} = 8,$$

which is in the interval. Evaluating $A(x)$ at $x = 8$ yields

$$A(8) = -8^2 + 16 \cdot 8 + 32 = -64 + 128 + 32 = 96.$$

iv. Conclusion. Comparing the three candidate values, $A(0) = 32$, $A(16) = 32$, and $A(8) = 96$, we see that the maximum possible area of a plot meeting the given requirements is 96 m^2. ■

[Total = 30]