Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
Section A, Trent University, Fall 2024

Solutions to the Final Examination
11:00-14:00 in the Gym on Tuesday, 10 December.

Instructions: Do both of parts **A** and **B**, and, if you wish, part **C**. Please show all your work, justify all your answers, and simplify these where you reasonably can. When you are asked to do k of n questions, only the first k that are not crossed out will be marked. *If you have a question, or are in doubt about something,* **ask!**

Aids: Any calculator, as long as it can't communicate with other devices; all sides of one letter- or A4-size sheet, with whatever you want written on it; your own brain.

Part A. Do all four (4) of **1**–**4**.

1. Compute $\dfrac{dy}{dx}$ as best you can in any four (4) of **a**–**f**. *[20 = 4 × 5 each]*

a. $y = \arctan\left(x^3\right)$ **b.** $y = x^2 e^{-x}$ **c.** $y = \ln\left(\sec(x) + \tan(x)\right)$

d. $y = \sec^3\left(\arctan(x)\right)$ **e.** $y = \dfrac{3+x^2}{4+x^2}$ **f.** $y = \left(\sin(x) + \cos(x)\right)^2$

Solutions. **a.** *Chain and Power Rules.*

$$\frac{dy}{dx} = \frac{d}{dx}\arctan\left(x^3\right) = \frac{1}{1+\left(x^3\right)^2} \cdot \frac{d}{dx}x^3 = \frac{3x^2}{1+x^6} \quad \square$$

b. *Product, Power, and Chain Rules.*

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2e^{-x}\right) = \left[\frac{d}{dx}x^2\right]e^{-x} + x^2\left[\frac{d}{dx}e^{-x}\right] = 2xe^{-x} + x^2e^{-x}\frac{d}{dx}(-x)$$
$$= 2xe^{-x} + x^2e^{-x}(-1) = x(2-x)e^{-x} \quad \square$$

c. *Chain Rule and algebra.*

$$\frac{dy}{dx} = \frac{d}{dx}\ln\left(\sec(x)+\tan(x)\right) = \frac{1}{\sec(x)+\tan(x)} \cdot \frac{d}{dx}\left(\sec(x)+\tan(x)\right)$$
$$= \frac{\sec(x)\tan(x)+\sec^2(x)}{\sec(x)+\tan(x)} = \frac{\sec(x)\left(\tan(x)+\sec(x)\right)}{\sec(x)+\tan(x)} = \sec(x) \quad \square$$

d. *Power and Chain Rules, and a trigonometric identity.*

$$\frac{dy}{dx} = \frac{d}{dx}\sec^3\left(\arctan(x)\right) = 3\sec^2\left(\arctan(x)\right) \cdot \frac{d}{dx}\arctan(x) = 3\sec^2\left(\arctan(x)\right) \cdot \frac{1}{1+x^2}$$
$$= \frac{3\sec^2\left(\arctan(x)\right)}{1+x^2} = \frac{3\left(1+\tan^2\left(\arctan(x)\right)\right)}{1+x^2} = \frac{3\left(1+x^2\right)}{1+x^2} = 3 \quad \square$$

e. *Quotient and Power Rules.*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{3+x^2}{4+x^2}\right) = \frac{\left[\frac{d}{dx}\left(3+x^2\right)\right]\left(4+x^2\right) - \left(3+x^2\right)\left[\frac{d}{dx}\left(4+x^2\right)\right]}{\left(4+x^2\right)^2} \\ &= \frac{2x\left(4+x^2\right) - \left(3+x^2\right)2x}{\left(4+x^2\right)^2} = \frac{8x+2x^3-6x-2x^3}{\left(4+x^2\right)^2} = \frac{2x}{\left(4+x^2\right)^2} \quad \square\end{aligned}$$

f. *Simplify first, then a little Chain Rule.*

$$\begin{aligned} y &= (\sin(x)+\cos(x))^2 = \sin^2(x) + 2\sin(x)\cos(x) + \cos^2(x) = 1+\sin(2x), \\ \text{so } \frac{dy}{dx} &= \frac{d}{dx}(1+\sin(2x)) = 0 + \cos(2x)\frac{d}{dx}(2x) = 2\cos(2x) \quad \square\end{aligned}$$

f. *Power and Chain Rules, and trigonometric identitites.*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\sin(x)+\cos(x))^2 = 2(\sin(x)+\cos(x))\frac{d}{dx}(\sin(x)+\cos(x)) \\ &= 2(\sin(x)+\cos(x))(\cos(x)-\sin(x)) = 2\left(\cos^2(x)-\sin^2(x)\right) = 2\cos(2x) \quad \blacksquare\end{aligned}$$

2. Evaluate any four (4) of the integrals **a**–**f**. *[20 = 4 × 5 each]*

a. $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx$ **b.** $\displaystyle\int_0^1 xe^{-x}\,dx$ **c.** $\displaystyle\int 2\ln(x)\,dx$

d. $\displaystyle\int_{-1}^1 (x+3)^3\,dx$ **e.** $\displaystyle\int \frac{x+1}{x^2-1}\,dx$ **f.** $\displaystyle\int_0^{\pi/2} \frac{\cos(x)}{1+\sin^2(x)}\,dx$

SOLUTIONS. **a.** *Substitution and Power Rule.* We will use the substitution $w = x^2+1$, so $dw = 2x\,dx$ and $x\,dx = \frac{1}{2}\,dw$.

$$\begin{aligned}\int \frac{x}{\sqrt{x^2+1}}\,dx &= \int \frac{1}{\sqrt{w}}\cdot\frac{1}{2}\,dw = \frac{1}{2}\int w^{-1/2}\,dw = \frac{1}{2}\cdot\frac{w^{1/2}}{1/2} + C = w^{1/2}+C \\ &= \left(x^2+1\right)^{1/2} + C = \sqrt{x^2+1}+C \quad \square\end{aligned}$$

b. *Integration by parts.* We will use integration by parts with $u = x$ and $v' = e^{-x}$, so $u' = 1$ and $v = -e^{-x}$.

$$\begin{aligned}\int_0^1 xe^{-x}\,dx &= x\left(-e^{-x}\right)\Big|_0^1 - \int_0^1 1\left(-e^{-x}\right)\,dx = 1\left(-e^{-1}\right) - 0\left(-e^{-0}\right) + \int_0^1 e^{-x}\,dx \\ &= -e^{-1} + 0 + \left(-e^{-1}\right)\Big|_0^1 = -e^{-1} + \left(-e^{-1}\right) - \left(-e^{-0}\right) = -2e^{-1}+1 \\ &= 1 - \frac{2}{e} \quad \square\end{aligned}$$

c. *Integration by parts.* We will use integration by parts with $u = \ln(x)$ and $v' = 2$, so $u' = \dfrac{1}{x}$ and $v = 2x$.

$$\begin{aligned}\int 2\ln(x)\,dx &= \ln(x)\cdot 2x - \int \frac{1}{x}\cdot 2x\,dx = 2x\ln(x) - \int 2\,dx = 2x\ln(x) - 2x + C \\ &= 2x\left(\ln(x) - 1\right) + C \quad \square\end{aligned}$$

d. *Substitution and Power Rule.* We will use the substitution $w = x + 3$, so $dw = dx$ and change the limits of integration as we go along: $\begin{array}{ccc} x & -1 & 1 \\ w & 2 & 4 \end{array}$.

$$\int_{-1}^{1} (x+3)^3\,dx = \int_2^4 u^3\,du = \left.\frac{u^4}{4}\right|_2^4 = \frac{4^4}{4} - \frac{2^4}{4} = \frac{256}{4} - \frac{16}{4} = \frac{240}{4} = 60 \quad \square$$

d. *Expand and then use the Power Rule.* Algebra first!

$$\begin{aligned}\int_{-1}^{1} (x+3)^3\,dx &= \int_{-1}^{1} \left(x^3 + 9x^2 + 27x + 27\right)\,dx = \left.\left(\frac{x^4}{4} + \frac{9x^3}{3} + \frac{27x^2}{2} + 27x\right)\right|_{-1}^{1} \\ &= \left(\frac{1^4}{4} + \frac{9\cdot 1^3}{3} + \frac{27\cdot 1^2}{2} + 27\cdot 1\right) \\ &\qquad - \left(\frac{(-1)^4}{4} + \frac{9(-1)^3}{3} + \frac{27(-1)^2}{2} + 27(-1)\right) \\ &= \frac{1}{4} + 3 + 13.5 + 27 - \frac{1}{4} + 3 - 13.5 + 27 = 60 \quad \square\end{aligned}$$

e. *Simplification, then Substitution.* We will end up using the substitution $w = x - 1$, so $dw = dx$.

$$\begin{aligned}\int \frac{x+1}{x^2-1}\,dx &= \int \frac{x+1}{(x+1)(x-1)}\,dx = \int \frac{1}{x-1}\,dx = \int \frac{1}{w}\,dw = \ln(w) + C \\ &= \ln(x-1) + C \quad \square\end{aligned}$$

f. *Substitution.* We will use the substitution $w = \sin(x)$, so $dw = \cos(x)\,dx$, and change the limits of integration as we go along: $\begin{array}{ccc} x & 0 & \pi/2 \\ w & 0 & 1 \end{array}$

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos(x)}{1+\sin^2(x)}\,dx &= \int_0^1 \frac{1}{1+w^2}\,dw = \arctan(w)\big|_0^1 = \arctan(1) - \arctan(0) \\ &= \frac{\pi}{4} - 0 = \frac{\pi}{4} \quad \blacksquare\end{aligned}$$

3. Do any four (4) of **a**–**g**. *[20 = 4 × 5 each]*

a. Compute $\lim_{x\to\infty} x^2 e^{-x}$.

b. Use the Right-Hand Rule to compute $\int_0^2 (x+1)\,dx$.

c. Use the ε–δ definition of limits to verify that $\lim_{x\to 2}(2x-3) = 1$.

d. Find the area of the region between $y = x^{1/3}$ and $y = x^3$, where $0 \le x \le 1$.

e. Determine whether $f(x) = \begin{cases} \frac{x^2}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous at $x = 0$ or not.

f. Find the volume of the solid obtained by revolving the region between the line $y = 1$ and the line $y = x$, for $0 \le x \le 1$, about the y-axis.

g. Use the limit definition of the derivative to compute $g'(x)$ for $g(x) = x^2 + x$.

SOLUTIONS. **a.** We will use l'Hôpital's Rule twice.

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2 \to \infty}{e^x \to \infty} = \lim_{x\to\infty} \frac{\frac{d}{dx}x^2}{\frac{d}{dx}e^x} = \lim_{x\to\infty} \frac{2x \to \infty}{e^x \to \infty} = \lim_{x\to\infty} \frac{\frac{d}{dx}x^2}{\frac{d}{dx}e^x}$$
$$= \lim_{x\to\infty} \frac{2 \to 2}{e^x \to \infty} = 0 \quad \square$$

b. The generic Right-Hand Rule formula for a definite integral is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{b-a}{n} \cdot f\left(a + i \cdot \frac{b-a}{n}\right) \right].$$

Here $a = 0$, $b = 2$, and $f(x) = x + 1$, so the Right-Hand Rule formula works out to:

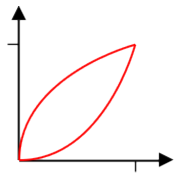
$$\int_0^2 (x+1)\,dx = \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{2-0}{n} \cdot f\left(0 + i \cdot \frac{2-0}{n}\right) \right] = \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{2}{n} \cdot \left(\frac{2i}{n} + 1\right) \right]$$
$$= \lim_{n\to\infty} \left[\frac{2}{n} \sum_{i=1}^{n} \left(\frac{2i}{n} + 1\right) \right] = \lim_{n\to\infty} \frac{2}{n} \left[\left(\sum_{i=1}^{n} \frac{2i}{n} \right) + \left(\sum_{i=1}^{n} 1 \right) \right]$$
$$= \lim_{n\to\infty} \frac{2}{n} \left[\left(\frac{2}{n} \sum_{i=1}^{n} i \right) + n \right] = \lim_{n\to\infty} \frac{2}{n} \left[\frac{2}{n} \cdot \frac{n(n+1)}{2} + n \right]$$
$$= \lim_{n\to\infty} \frac{2}{n} \left[(n+1) + n \right] = \lim_{n\to\infty} \frac{2}{n} \left[2n + 1 \right] = \lim_{n\to\infty} \left[4 + \frac{2}{n} \right] = 4 + 0 = 4 \quad \square$$

c. To verify that $\lim_{x\to 2}(2x-3) = 1$ using ε–δ definition of limits, we need to check that for any $\varepsilon > 0$ we can find a $\delta > 0$ such that if $|x-2| < \delta$, then $|(2x-3) - 1| < \varepsilon$. As usual, we try to reverse-engineer the required δ from the condition ε is to satisfy. Suppose that $\varepsilon > 0$. Then:

$$|(2x-3)-1| < \varepsilon \iff |2x-4| < \varepsilon \iff 2|x-2| < \varepsilon \iff |x-2| < \frac{\varepsilon}{2}$$

If we now set $\delta = \frac{\varepsilon}{2}$, then $|x-2| < \delta$ will imply that $|(2x-3)-1| < \varepsilon$ because every step above is reversible.

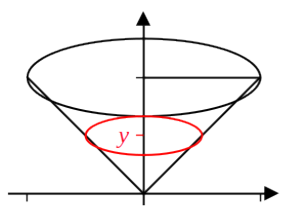
It follows that $\lim_{x \to 2} (2x-3) = 1$ by the ε–δ definition of limits. $\square$



d. The two curves intersect when $x^3 = x^{1/3}$, that is, when $x^9 = \left(x^3\right)^3 = \left(x^{1/3}\right)^3 = x$, which is satisfied by the real numbers -1, 0, and 1. Only the latter two matter here, as we are told that $0 \le x \le 1$ for the given region. It's easy to check that between 0 and 1, $y = x^{1/3}$ is above $y = x^3$; for one example, at $x = \frac{1}{2} = 0.5$, $0.5^3 = 0.125 < 0.7937 \approx 0.5^{1/3}$. It follows that the area of the region between the two curves is given by:

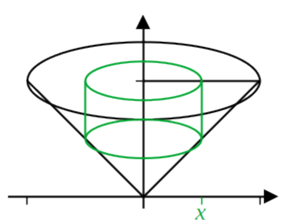
$$\begin{aligned}\text{Area} &= \int_0^1 \left(x^{1/3} - x^3\right)\,dx = \left(\frac{x^{4/3}}{4/3} - \frac{x^4}{4}\right)\Bigg|_0^1 = \left(\frac{3}{4}\cdot 1^{4/3} - \frac{1^4}{4}\right) - \left(\frac{3}{4}\cdot 0^{4/3} - \frac{0^4}{4}\right) \\ &= \frac{3}{4} - \frac{1}{4} - 0 = \frac{2}{4} = \frac{1}{2} \quad \square\end{aligned}$$

e. We simplify things a little bit first. Whenever $x \neq 0$, we have $\frac{x^2}{|x|} = \frac{|x|^2}{|x|} = |x|$, and so $f(x) = \begin{cases} \frac{x^2}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases} = \begin{cases} |x| & x \neq 0 \\ 0 & x = 0 \end{cases} = \begin{cases} -x & x < 0 \\ 0 & x = 0 \\ x & x > 0 \end{cases}$. As $\lim_{x\to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$, we have that $\lim_{x\to 0} f(x) = 0 = f(0)$. Thus $f(x)$ is continuous at $x = 0$ by the definition of continuity. $\square$



f. *Disk/washer method.* Since we revolved the region about the y-axis the disk cross-sections of the solid are perpendicular to the y-axis, so we should use y as the basic variable. The disk at level y has radius $x - 0 = x = y$ and hence area $\pi r^2 = \pi y^2$. Since we also have $0 \le y \le 1$ for the given region, the volume of the solid is given by:

$$V = \int_0^1 \pi r^2\,dy = \int_0^1 \pi y^2\,dy = \pi \cdot \frac{y^3}{3}\Bigg|_0^1 = \pi \cdot \frac{1^3}{3} - \pi \cdot \frac{0^3}{3} = \frac{\pi}{3} \quad \square$$



f. *Cylindrical shell method.* Since we revolved the region about the y-axis the cylindrical cross-sections of the solid are perpendicular to the x-axis, so we should use x as the basic variable. The cylinder at x has radius $r = x - 0 = x$ and height $h = 1 - y = 1 - x$ and so area $2\pi r h = 2\pi x(1-x)$. Since we are given that $0 \le x \le 1$ for the given region, the volume of the solid is given by:

$$
\begin{aligned}
V &= \int_0^1 2\pi r h \, dx = \int_0^1 2\pi x(1-x)\, dx = 2\pi \int_0^1 \left(x - x^2\right) \, dx = 2\pi \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^1 \\
&= 2\pi \left(\frac{1^2}{2} - \frac{1^3}{3}\right) - 2\pi \left(\frac{0^2}{2} - \frac{0^3}{3}\right) = 2\pi \cdot \frac{1}{6} - 2\pi \cdot 0 = \frac{\pi}{3} \quad \square
\end{aligned}
$$

g. By the limit definition of the derivative,

$$
\begin{aligned}
g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{\left[(x+h)^2 + (x+h)\right] - \left[x^2 + x\right]}{h} \\
&= \lim_{h \to 0} \frac{\left[x^2 + 2xh + h^2 + x + h\right] - \left[x^2 + x\right]}{h} = \lim_{h \to 0} \frac{2xh + h^2 + h}{h} \\
&= \lim_{h \to 0} (2x + h + 1) = 2x + 0 + 1 = 2x + 1. \quad \blacksquare
\end{aligned}
$$

4. Find the domain, intercepts, vertical and horizontal asymptotes, intervals of increase and decrease, maximum and minimum points, intervals of concavity, and inflection points of $f(x) = x^2 e^{-x}$, and sketch its graph based on this information. *[12]*

Solution. We run through the indicated checklist ...

i. Domain. x^2 and e^{-x} are both defined (and continuous and differentiable) for all x, and hence so is their product. Thus $f(x) = x^2 e^{-x}$ has domain $\mathbb{R} = (-\infty, \infty)$.

ii. Intercepts. $f(0) = 0^2 e^{-0} = 0 \cdot 1 = 0$, so the y-intercept is $y = 0$. Since $e^{-x} > 0$ for all x, $f(x) = 0$ exactly when $x^2 = 0$, *i.e.* exactly when $x = 0$, so the only x-intercept is $x = 0$. Note that the only x-intercept is also the y-intercept.

iii. Vertical asymptotes. Since $f(x) = x^2 e^{-x}$ is defined and continuous for all x, it cannot have any vertical asymptotes.

iv. Horizontal asymptotes. We take the limits as $x \to \pm\infty$, with some help from l'Hôpital's Rule, and see what happens:

$$
\begin{aligned}
\lim_{x \to -\infty} x^2 e^{-x} &= \lim_{x \to -\infty} \frac{x^2 \to +\infty}{e^x \to 0^+} = +\infty \\
\lim_{x \to +\infty} x^2 e^{-x} &= \lim_{x \to +\infty} \frac{x^2 \to +\infty}{e^x \to +\infty} = \lim_{x \to +\infty} \frac{\frac{d}{dx} x^2}{\frac{d}{dx} e^x} = \lim_{x \to +\infty} \frac{2x \to +\infty}{e^x \to +\infty} \\
&= \lim_{x \to +\infty} \frac{\frac{d}{dx} 2x}{\frac{d}{dx} e^x} = \lim_{x \to +\infty} \frac{2 \to 2}{e^x \to +\infty} = 0^+
\end{aligned}
$$

$y = f(x)$ thus has a horizontal asymptote of $y = 0$ to the right (*i.e.* as $x \to +\infty$), which it approaches from above, but does not have one to the left (*i.e.* as $x \to -\infty$).

v. Increase/decrease & max/min. We first compute $f'(x)$, using the Product, Power, and Chain Rules:

$$
f'(x) = \frac{d}{dx}\left(x^2 e^{-x}\right) = \left[\frac{d}{dx} x^2\right] e^{-x} + x^2 \left[\frac{d}{dx} e^{-x}\right] = 2x e^{-x} + x^2 e^{-x} \frac{d}{dx}(-x) = x(2-x)e^{-x}
$$

Since $e^{-x} > 0$ for all x, $f'(x) = x(2-x)e^{-x}$ is positive, zero, or negative exactly when $x(2-x)$ is. When $x < 0$, $2 - x > 0$, so $x(2-x) < 0$; when $x = 0$, $x(2-x) = 0$; when $0 < x < 2$, $2 - x > 0$, so $x(2-x) > 0$; when $x = 2$, $x(2-x) = 0$; and when $x > 2$, $x > 0$ and $2 - x < 0$, so $x(2-x) < 0$. It follows that $f(x)$ is decreasing on $(-\infty, 0)$, increasing on $(0, 2)$, and decreasing again on $(2, +\infty)$, and thus has a local (and absolute) minimum at $x = 0$ and a local maximum (which is not absolute) at $x = 2$. We summarize all this in the following table:

x	$(-\infty, 0)$	0	$(0, 2)$	2	$(2, +\infty)$
$f'(x)$	$-$	0	$+$	0	$+$
$f(x)$	$\downarrow$	min	$\uparrow$	max	$\downarrow$

vi. Concavity and inflection points. We first compute $f''(x)$, using the Product, Power, and Chain Rules:

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(x(2-x)e^{-x}\right) = \frac{d}{dx}\left(\left(2x - x^2\right)e^{-x}\right) = \left[\frac{d}{dx}\left(2x - x^2\right)\right]e^{-x} + \left(2x - x^2\right)\left[e^{-x}\right] \\ &= (2 - 2x)e^{-x} + \left(2x - x^2\right)e^{-x}\frac{d}{dx}(-x) = (2 - 2x)e^{-x} + \left(2x - x^2\right)e^{-x}(-1) \\ &= \left(2 - 2x + x^2 - 2x\right)e^{-x} = \left(x^2 - 4x + 2\right)e^{-x} = \left(x - \left(2 - \sqrt{2}\right)\right)\left(x - \left(2 + \sqrt{2}\right)\right) \end{aligned}$$

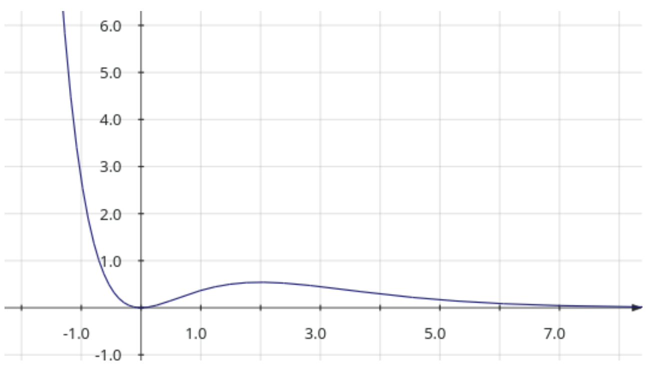
The last step follows because, using the quadratic formula, $x^2 - 4x + 2 = 0$ exactly when

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{4 \pm \sqrt{16 - 8}}{2} = \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}.$$

This also means that $f''(x) = 0$ exactly when $x = 2 \pm \sqrt{2}$. When $x < 2 - \sqrt{2}$, $x - \left(2 - \sqrt{2}\right) < 0$ and $x - \left(2 + \sqrt{2}\right) < 0$, so $f''(x) > 0$; when $2 - \sqrt{2} < x < 2 + \sqrt{2}$, $x - \left(2 - \sqrt{2}\right) > 0$ and $x - \left(2 + \sqrt{2}\right) < 0$, so $f''(x) < 0$; and when $x > 2 + \sqrt{2}$, $x - \left(2 - \sqrt{2}\right) > 0$ and $x - \left(2 + \sqrt{2}\right) > 0$, so $f''(x) > 0$. This means that $y = f(x)$ is concave up when $x < 2 - \sqrt{2}$, concave down when $2 - \sqrt{2} < x < 2 + \sqrt{2}$, and concave up again when $x > 2 + \sqrt{2}$, and thus has inflection points at $x = 2 \pm \sqrt{2}$. We summarize all this in the following table:

x	$\left(-\infty, 2 - \sqrt{2}\right)$	$2 - \sqrt{2}$	$\left(2 - \sqrt{2}, 2 + \sqrt{2}\right)$	$2 + \sqrt{2}$	$\left(2 + \sqrt{2}, +\infty\right)$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\smile$	infl.	$\frown$	infl.	$\smile$

vii. Graph. Cheating just a bit, we have a computer program called `kmplot` draw the graph for us:



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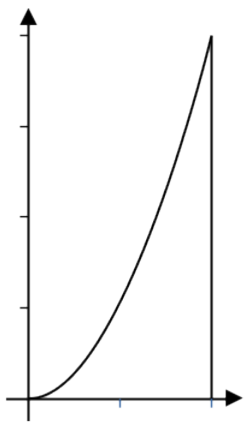
Part B. Do any two (2) of **5–7**. *[28 = 2 × 14 each]*

5. The region below $y = x^2$ and above $y = 0$, where $0 \leq x \leq 2$, is revolved about the line $x = -1$, making a solid of revolution.

a. Sketch the region. *[1]* **b.** Find the area of the region. *[3]*

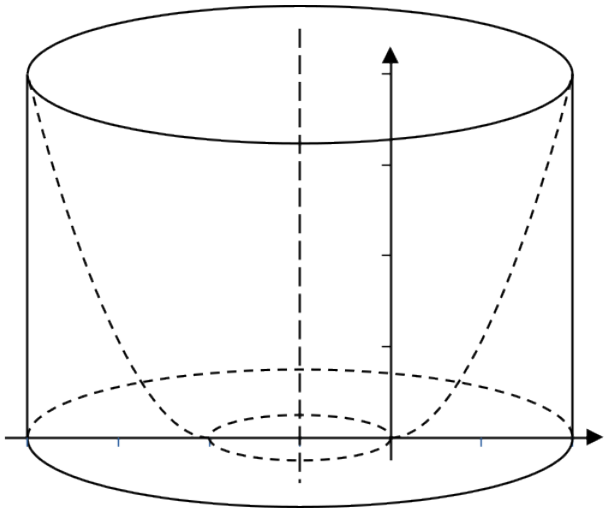
c. Sketch the solid. *[1]* **d.** Find the volume of the solid. *[9]*

SOLUTIONS. **a.**



□

c.

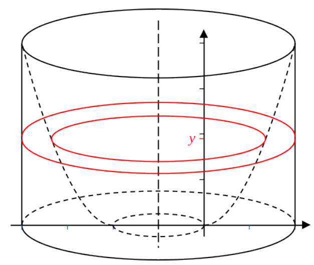


□

b. The area of the region is

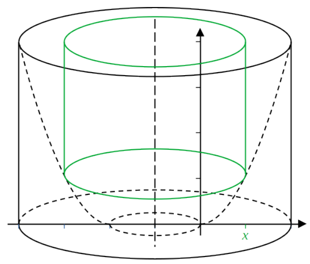
$$A = \int_0^2 \left(x^2 - 0\right)\, dx = \int_0^2 x^2\, dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3} - 0 = \frac{8}{3}. \quad \square$$

d. *Disk/washer method.* Since we revolved the region about a vertical line, washer cross-sections of the solid are perpendicular to the y-axis, so we use y as the basic variable. The washer at level y has outer radius $R = 2 - (-1) = 3$ (since the right edge of the region is the line $x = 2$) and inner radius $r = x - (-1)$ $= x + 1 = 1 + \sqrt{y}$ (as the left edge is the parabola $y = x^2$). It therefore has area equal to $\pi\left(R^2 - r^2\right) = \pi\left(3^2 - \left(1 + \sqrt{y}\right)^2\right)$ $= \pi\left(9 - \left(1 + 2\sqrt{y} + y\right)\right) = \pi\left(8 - 2\sqrt{y} - y\right)$. Note also that $0 \leq y \leq 4$ over the given region. The volume of the solid is then given by:



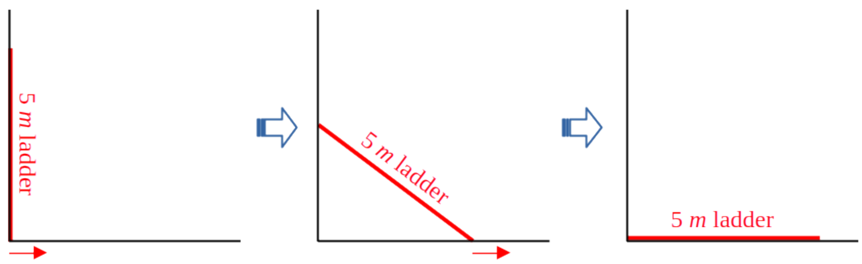
$$\begin{aligned}
V &= \int_0^4 \pi\left(R^2 - r^2\right)\, dy = \int_0^4 \pi\left(8 - 2\sqrt{y} - y\right)\, dy = \pi \int_0^4 \left(8 - 2y^{1/2} - y\right)\, dy \\
&= \pi\left.\left(8y - \frac{2y^{3/2}}{3/2} - \frac{y^2}{2}\right)\right|_0^4 = \pi\left.\left(8y - \frac{4y\sqrt{y}}{3} - \frac{y^2}{2}\right)\right|_0^4 \\
&= \pi\left(8 \cdot 4 - \frac{4 \cdot 4\sqrt{4}}{3} - \frac{4^2}{2}\right) - \pi\left(8 \cdot 0 - \frac{4 \cdot 0\sqrt{0}}{3} - \frac{0^2}{2}\right) = \pi\left(32 - \frac{32}{3} - \frac{16}{2}\right) - \pi \cdot 0 \\
&= \pi\left(32 - \frac{32}{3} - 8\right) = \pi\left(24 - \frac{32}{3}\right) = \pi\left(\frac{72}{3} - \frac{32}{3}\right) = \frac{40\pi}{3} \quad \square
\end{aligned}$$

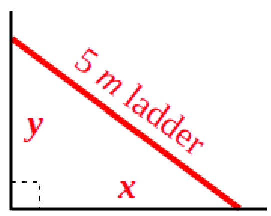
d. *Cylindrical shell method.* Since we revolved the region about a vertical line, the cylindrical cross-sections of the solid are perpendicular to the x-axis, so we use x as the basic variable. The cylinder at x has radius $r = x - (-1) = x + 1$ and height $h = y - 0 = x^2$, and therefore has area $2\pi rh = 2\pi(x+1)x^2 = 2\pi\left(x^3 + x^2\right)$. Recall that we are given that $0 \leq x \leq 2$ for the original region. Thus the volume of the solid is given by:



$$\begin{aligned}
V &= \int_0^2 2\pi rh = \int_0^2 2\pi\left(x^3 + x^2\right)\,dx = 2\pi\left(\frac{x^4}{4} + \frac{x^3}{3}\right)\Bigg|_0^2 \\
&= 2\pi\left(\frac{2^4}{4} + \frac{2^3}{3}\right) - 2\pi\left(\frac{0^4}{4} + \frac{0^3}{3}\right) = 2\pi\left(4 + \frac{8}{3}\right) - 2\pi \cdot 0 \\
&= 2\pi\left(\frac{12}{3} + \frac{8}{3}\right) - 0 = 2\pi \cdot \frac{20}{3} = \frac{40\pi}{3} \quad \blacksquare
\end{aligned}$$

6. A 5 m long ladder is flush up against a vertical wall at first. Its bottom then slides on the horizontal floor away from the wall, the top and bottom of the ladder maintaining contact with the wall and floor, respectively, until the ladder rests on the floor. What is the maximum area of the triangle made by the wall, floor, and ladder during this process? *[14]*





Solution. The triangle made by the wall, floor, and ladder at any given instant is a right triangle, because the wall and floor are perpendicular to each other, with the ladder as its hypotenuse. Let y be the height of this triangle and x its base, so $x^2 + y^2 = 25$ by the Pythagorean Theorem. We can solve this equation for y in terms of x: $y^2 = 25 - x^2$, so $y = \sqrt{25 - x^2}$ – we use the positive root because y is a length and hence $y \geq 0$. Note that the possible values of x are between 0 (at the start of the slide) and 5 (at the end of the slide), inclusive. We can now express the area of the triangle as a function of x: $A(x) = \frac{1}{2}xy = \frac{1}{2}x\sqrt{25 - x^2}$.

We maximize $A(x)$ for $0 \leq x \leq 5$. First, with the help of the Product, Power, and Chain Rules, we compute $A'(x)$:

$$
\begin{aligned}
A'(x) &= \frac{d}{dx}\left(\frac{1}{2}x\sqrt{25-x^2}\right) = \frac{1}{2}\left[\frac{d}{dx}x\right]\sqrt{25-x^2} + \frac{1}{2}x\left[\frac{d}{dx}\sqrt{25-x^2}\right] \\
&= \frac{1}{2}\cdot 1\sqrt{25-x^2} + \frac{1}{2}x\cdot\frac{1}{2\sqrt{25-x^2}}\cdot\left[\frac{d}{dx}\left(25-x^2\right)\right] \\
&= \frac{1}{2}\sqrt{25-x^2} + \frac{1}{2}\cdot\frac{x}{2\sqrt{25-x^2}}(-2x) = \frac{1}{2}\left[\sqrt{25-x^2} - \frac{x^2}{\sqrt{25-x^2}}\right]
\end{aligned}
$$

Second, we find the critical points of $A(x)$ in the interval $[0,5]$:

$$
\begin{aligned}
A'(x) = 0 &\implies \sqrt{25-x^2} - \frac{x^2}{\sqrt{25-x^2}} = 0 \implies \left(\sqrt{25-x^2}\right)^2 - x^2 = 0 \\
&\implies 25 - x^2 - x^2 = 0 \implies 25 - 2x^2 = 0 \implies x^2 = \frac{25}{2} \implies x = \pm\frac{5}{\sqrt{2}}
\end{aligned}
$$

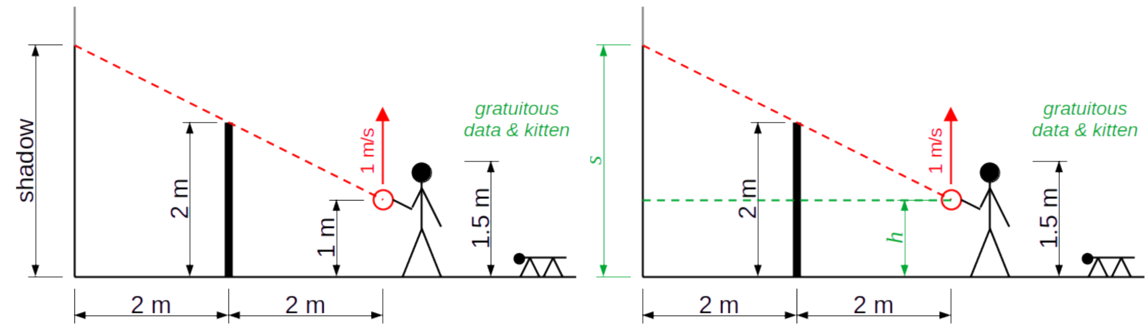
$x = -\frac{5}{\sqrt{2}}$ is not between 0 and 5, while $x = +\frac{5}{\sqrt{2}}$ is, so we only need to consider the latter critical point.

Third, we compare the areas at the endpoints with the area at the critical point to see which is the largest.

$$
\begin{aligned}
A(0) &= \frac{1}{2}\cdot 0\cdot\sqrt{25-0^2} = 0 \\
A(5) &= \frac{1}{2}\cdot 5\cdot\sqrt{25-5^2} = \frac{5}{2}\cdot 0 = 0 \\
A\left(\frac{5}{\sqrt{2}}\right) &= \frac{1}{2}\cdot\frac{5}{\sqrt{2}}\cdot\sqrt{25-\left(\frac{5}{\sqrt{2}}\right)^2} = \frac{1}{2}\cdot\frac{5}{\sqrt{2}}\cdot\sqrt{25-\frac{25}{2}} \\
&= \frac{1}{2}\cdot\frac{5}{\sqrt{2}}\cdot\sqrt{\frac{25}{2}} = \frac{1}{2}\cdot\frac{5}{\sqrt{2}}\cdot\frac{5}{\sqrt{2}} = \frac{25}{4}
\end{aligned}
$$

Thus the maximum possible area of the triangle formed by the ladder, wall, and floor is $\frac{25}{4} = 6.25$. (That area is in m^2, if you care about the units, since the length of the ladder was given in m.) ■

7. A vertical post 2 m tall stands on level ground 2 m from a vertical wall. Stick Person stands with a lantern that is 2 m from the post. Stick raises the lantern vertically at 1 m/s. How is the length of the shadow, as cast by the post onto the wall by the lantern's light, changing at the instant that the lantern is 1 m above the ground? *[14]*



Solution. Let s be the length of the shadow cast by the post on the wall, and let h be the height the lantern is above the ground, as in the modified diagram above right. We are told that $\frac{dh}{dt} = 1\ m/s$ and asked to compute $\frac{ds}{dt}$ at the instant that $h = 1\ m$.

Consider the two right triangles formed by the beam from the lantern, the horizontal line $h\ m$ above the ground, and the portions of the wall and post, respectively, between the beam and the horizontal line. These triangles are similar, *i.e.* they have the same proportions, so the ratios of their heights to their bases must be the same: $\frac{s-h}{4} = \frac{2-h}{2}$. It then follows that $s - h = 4 - 2h$, so $s = 4 - h$, and from this that $\frac{ds}{dt} = -\frac{dh}{dt} = -1\ m/s$.

Thus the length of the shadow is changing by $-1\ m/s$, *i.e.* it is decreasing at rate of $1\ m/s$. Note that the information that $h = 1\ m$ at the given instant isn't necessary here; you only need to know that $\frac{dh}{dt} = 1\ m/s$. ■

[Total = 100]

Part C. *Bonus points!* Do one or both of $\mathbf{2^3}$ and $\mathbf{3^2}$.

$\mathbf{2^3}$**.** Write an original haiku touching on calculus or mathematics in general. *[1]*

What is a haiku?

seventeen in three:
five and seven and five of
syllables in lines

$\mathbf{3^2}$**.** Verify that $\ln(\sec(x) - \tan(x)) = -\ln(\sec(x) + \tan(x))$. *[1]*

You're on your own for the bonus problems!

APOLOGIES FOR ALL THE GLITCHES THIS TERM. HAVE A GOOD BREAK!