Trigonometric Identities, Limits, Derivatives, and Integrals
A Very Brief Summary

In general, we'll only deal with four trigonometric functions, $\sin(x)$ (sine), $\cos(x)$ (cosine), $\tan(x) = \frac{\sin(x)}{\cos(x)}$ (tangent), and $\sec(x) = \frac{1}{\cos(x)}$ (secant). The remaining two standard trigonometric functions, $\cot(x)) = \frac{\cos(x)}{\sin(x)}$ (cotangent) and $\csc(x) = \frac{1}{\sin(x)}$ (cosecant), don't come up nearly as often and are usually looked up when they do come up ...

0. A small set of trigonometric identities

- $\sin^2(x) + \cos^2(x) = 1$
  [Often used in the form $\cos^2(x) = 1 - \sin^2(x)$ or $\sin^2(x) = 1 - \cos^2(x)$.]
- $1 + \tan^2(x) = \sec^2(x)$
  [Sometimes used in the form $\sec^2(x) - 1 = \tan^2(x)$.]
- $\sin(2x) = 2\sin(x)\cos(x)$
- $$\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x)\\ &= 2\cos^2(x) - 1\\ &= 1 - 2\sin^2(x)\end{aligned}$$
  [Sometimes used in the form $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$ or $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$.]
- The double angle identities above are special cases of the addition identities $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ and $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$.

It is also useful to keep in mind that:

- $\sin(x)$, $\cos(x)$, and $\sec(x)$ are *periodic* with period 2π: for any real number x and any integer n, $\sin(x + 2n\pi) = \sin(x)$, $\cos(x + 2n\pi) = \cos(x)$, and $\sec(x + 2n\pi) = \sec(x)$.
- $\tan(x)$ is periodic with period π: for any real number x and any integer n, $\tan(x + n\pi) = \tan(x)$.
- $\sin(x)$ and $\tan(x)$ are *odd* functions, $\sin(-x) = -\sin(x)$ and $\tan(-x) = -\tan(x)$ for all x, while $\cos(x)$ and $\sec(x)$ are *even* functions, $\cos(-x) = \cos(x)$ and $\sec(-x) = \sec(x)$ for all x.
- Phase shifts are fun: $\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$, $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$, $\sin(x \pm \pi) = -\sin(x)$, and $\cos(x \pm \pi) = -\cos(x)$, for all x. (You can have some fun working out what this means for $\tan(x)$ and $\sec(x)$. :-))

1. The key trigonometric limits

- If $f(x)$ is any of the trigonometric functions and it is defined at $x = a$, then it is continuous at $x = a$, *i.e.* $\lim_{x\to a} f(x) = f(a)$.
- $\tan(x)$ has asymptotes at $x = n\pi + \frac{\pi}{2}$ for each integer n. If $a = n\pi + \frac{\pi}{2}$, then $\lim_{x\to a^-} \tan(x) = \infty$ and $\lim_{x\to a^+} \tan(x) = -\infty$.
- $\sec(x)$ has asymptotes at $x = n\pi + \frac{\pi}{2}$ for each integer n. If $a = n\pi + \frac{\pi}{2}$, then $\lim_{x\to a^-} \sec(x) = \infty$ and $\lim_{x\to a^+} \sec(x) = -\infty$ if n is even, and $\lim_{x\to a^-} \sec(x) = -\infty$ and $\lim_{x\to a^+} \sec(x) = \infty$ if n is odd.
- $\displaystyle\lim_{h\to 0} \frac{\sin(h)}{h} = 1$ and $\displaystyle\lim_{h\to 0} \frac{\cos(h) - 1}{h} = 0$.

2. The key trigonometric derivatives

- $\dfrac{d}{dx}\sin(x) = \cos(x)$ and $\dfrac{d}{dx}\cos(x) = -\sin(x)$.
- $\dfrac{d}{dx}\tan(x) = \sec^2(x)$ and $\dfrac{d}{dx}\sec(x) = \sec(x)\tan(x)$.

4. Some trigonometric integral reduction formulas

The following formulas can each be obtained by a judicious use of trigonometric identities, algebra, integration by parts, and substitution. So long as $n \geq 2$, we have:

- $\displaystyle\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$
- $\displaystyle\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$
- $\displaystyle\int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx$
- $\displaystyle\int \sec^n(x)\,dx = \frac{1}{n-1}\tan(x)\sec^{n-2}(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$
- Just for fun – one usually looks this up as necessary – if we also have $k \geq 2$, then:

$$\begin{aligned}\int \sin^k(x)\cos^n(x)\,dx &= -\frac{\sin^{k-1}(x)\cos^{n+1}(x)}{k+n} + \frac{k-1}{k+n}\int \sin^{k-2}(x)\cos^n(x)\,dx \\ &= +\frac{\sin^{k+1}(x)\cos^{n-1}(x)}{k+n} + \frac{n-1}{k+n}\int \sin^k(x)\cos^{n-2}(x)\,dx\end{aligned}$$

For real obscurity, try to find or compute the corresponding formulas for integrands with mixed $\sec(x)$ and $\tan(x)$, not to mention the various reduction formulas involving $\csc(x)$ and/or $\cot(x)$.