Trent University, Fall 2023

Solutions to MATH 1110H-B Test

Monday, 30 October

Time: 60 minutes

Name: Minus Zero

Student Number: -0000000

Question	Mark	
1	___	
2	___	
3 or 4	___	
Total	___	/30

Instructions

- *Show all your work.* Legibly, please! Simplify where you reasonably can.
- *If you have a question, ask it!*
- Use the back sides of all the pages for rough work or extra space.
- You may use a calculator and (all sides of) one letter- or A4-size aid sheet.
- If you do more than the minimum number of parts or questions, only the first ones the marker finds will be marked. Cross out anything you do not want marked.

1. Do any *two* (2) of parts **a**–**c**. *[10 = 2 × 5 each]*

a. Use the ε–δ definition of limits to verify that $\lim_{x \to 2} (2x-5) = -1$.

b. Compute $\lim_{t \to 0} \frac{4t}{e^{2t}-1}$.

c. Use the limit definition of the derivative to find $\frac{d}{dx}\left(x^2 - x + 4\right)$.

SOLUTIONS. **a.** We need to check that for every $\varepsilon > 0$ there is some $\delta > 0$ such that if $|x-2| < \delta$, then $|(2x-5)-(-1)| < \varepsilon$.

Suppose we are given an $\varepsilon > 0$. As usual, we try to work out a suitable $\delta > 0$ by reverse-engineering:

$$\begin{aligned} |(2x-5)-(-1)| < \varepsilon &\iff |2x-4| < \varepsilon \\ &\iff 2\,|x-2| < \varepsilon \\ &\iff |x-2| < \frac{\varepsilon}{2} \end{aligned}$$

We set $\delta = \frac{\varepsilon}{2}$. Since all of the steps above are reversible, if we have $|x-2|| < \delta = \frac{\varepsilon}{2}$, then we must also have $|(2x-5)-(-1)| < \varepsilon$. Thus $\lim_{x \to 2} (2x-5) = -1$ by the ε–δ definition of limits. $\square$

b. Let $h = 2t$; then $h \to 0$ as $t \to 0$. Recall from class or textbook that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$. It follows that:

$$\begin{aligned} \lim_{t \to 0} \frac{4t}{e^{2t}-1} &= \lim_{h \to 0} \frac{2h}{e^h - 1} = 2 \lim_{h \to 0} \frac{h}{e^h - 1} \\ &= 2 \lim_{h \to 0} \frac{1}{\frac{e^h-1}{h}} = 2 \frac{1}{\lim_{h \to 0} \frac{e^h-1}{h}} = 2 \cdot \frac{1}{1} = 2 \quad \square \end{aligned}$$

c. We apply the limit definition of the derivative to $y = x^2 - x + 4$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(x^2 - x + 4\right) = \lim_{h \to 0} \frac{\left((x+h)^2 - (x+h) + 4\right) - \left(x^2 - x + 4\right)}{h} \\ &= \lim_{h \to 0} \frac{\left(x^2 + 2hx + h^2 - x - h + 4\right) - \left(x^2 - x + 4\right)}{h} = \lim_{h \to 0} \frac{2hx + h^2 - h}{h} \\ &= \lim_{h \to 0} \frac{h\,(2x + h - 1)}{h} = \lim_{h \to 0} (2x + h - 1) = 2x + 0 - 1 = 2x - 1 \quad \blacksquare \end{aligned}$$

2. Find $\dfrac{dy}{dx}$ as best you can in any *two* (2) of parts **a**–**c**. *[10 = 2 × 5 each]*

a. $e^{y+x} = e^x$ **b.** $y = \left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}+1\right)$ **c.** $y = \sin\left(x + \dfrac{1}{x}\right)$

SOLUTIONS. **a.** *Differentiate, then simplify.* Chain Rule and algebra:

$$e^{y+x} = e^x \implies \frac{d}{dx}\left(e^{y+x}\right) = \frac{d}{dx}e^x \implies e^{y+x}\cdot\frac{d}{dx}(x+y) = e^x \implies e^{y+x}\left(1+\frac{dy}{dx}\right) = e^x$$
$$\implies 1 + \frac{dy}{dx} = \frac{e^x}{e^{y+x}} = \frac{e^x}{e^y e^x} = \frac{1}{e^y} = e^{-y} \implies \frac{dy}{dx} = e^{-y} - 1$$

This is a correct (and acceptable marks-wise) answer, but one can do better ... □

a. *Simplify, then differentiate.* Observe that we have $e^y e^x = e^{y+x} = e^x$, and as $e^x > 0$ for all x, it follows that $e^y = 1$ for all x, so $y = \ln(1) = 0$ for all x. Thus $\dfrac{dy}{dx} = \dfrac{d}{dx}0 = 0$. □

b. *Differentiate, then simplify.* Product, Power, and Chain Rules, plus algebra:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left[\left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}+1\right)\right] = \frac{d}{dx}\left[\left((x+1)^{1/2}-1\right)\left((x+1)^{1/2}+1\right)\right] \\
&= \left[\frac{d}{dx}\left((x+1)^{1/2}-1\right)\right]\left((x+1)^{1/2}+1\right) + \left((x+1)^{1/2}-1\right)\left[\frac{d}{dx}\left((x+1)^{1/2}+1\right)\right] \\
&= \left[\frac{1}{2}(x+1)^{-1/2}\frac{d}{dx}(x+1) - 0\right]\left((x+1)^{1/2}+1\right) \\
&\qquad + \left((x+1)^{1/2}-1\right)\left[\frac{1}{2}(x+1)^{-1/2}\frac{d}{dx}(x+1) + 0\right] \\
&= \frac{1}{2}(x+1)^{-1/2}\left((x+1)^{1/2}+1\right) + \left((x+1)^{1/2}-1\right)\frac{1}{2}(x+1)^{-1/2} \\
&= \frac{1}{2}(x+1)^{-1/2}\,(x+1)^{1/2} + \frac{1}{2}(x+1)^{-1/2} + \frac{1}{2}(x+1)^{-1/2}(x+1)^{1/2} - \frac{1}{2}(x+1)^{-1/2} \\
&= \frac{1}{2} + \frac{1}{2} = 1 \quad \square
\end{aligned}$$

b. *Simplify, then differentiate.* Multiply out the expression for y and simplify:

$$y = \left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}+1\right) = \left(\sqrt{x+1}\right)^2 + 1\sqrt{x+1} - 1\sqrt{x+1} - 1^2 = x+1-1 = x$$

Thus $\dfrac{dy}{dx} = \dfrac{d}{dx}x = 1$. □

c. Chain Rule and Power Rule – no cheap simplifications here ...

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\sin\left(x+\frac{1}{x}\right) = \frac{d}{dx}\sin\left(x+x^{-1}\right) = \cos\left(x+x^{-1}\right)\frac{d}{dx}\left(x+x^{-1}\right) \\
&= \cos\left(x+x^{-1}\right)\cdot\left(1+(-1)x^{-2}\right) = \left(1-\frac{1}{x^2}\right)\cos\left(x+\frac{1}{x}\right) \quad \blacksquare
\end{aligned}$$

Do *one* (1) of questions **3** or **4**. *[10]*

3. Suppose $y = a\cos(x) + b\sin(x)$ for some constants a and b.

i. Show that this function satisfies the differential equation $\dfrac{d^2y}{dx^2} = -y$. *[6]*

ii. Determine the constants a and b if it is known that $y = 1$ when $x = 0$ and also when $x = \dfrac{\pi}{2}$. *[4]*

SOLUTIONS. *i.* We differentiate twice to check whether $\dfrac{d^2y}{dx^2} = -y$.

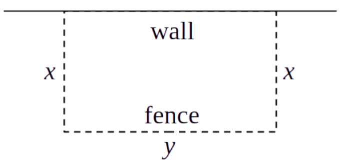
$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d^2}{dx^2}\left(a\cos(x) + b\sin(x)\right) = \frac{d}{dx}\left[\frac{d}{dx}\left(a\cos(x) + b\sin(x)\right)\right] \\ &= \frac{d}{dx}\left[-a\sin(x) + b\cos(x)\right] = -a\cos(x) - b\sin(x) = -\left(a\cos(x) + b\sin(x)\right) = -y\end{aligned}$$

The function $y = a\cos(x) + b\sin(x)$ therefore does satisfy the equation $\dfrac{d^2y}{dx^2} = -y$. $\square$

ii. Recall that $\cos(0) = \sin\left(\frac{\pi}{2}\right) = 1$ and $\cos\left(\frac{\pi}{2}\right) = \sin(0) = 0$.

When $x = 0$, $1 = y = a\cos(0) + b\sin(0) = a \cdot 1 + b \cdot 0 = a$, so $a = 1$. Similarly, when $x = \dfrac{\pi}{2}$, $1 = y = a\cos\left(\frac{\pi}{2}\right) + b\sin\left(\frac{\pi}{2}\right) = a \cdot 0 + b \cdot 1 = b$, so $b = 1$ too. $\blacksquare$

4. A rectangular plot next to a wall is to be fenced off, using the wall as one side of the plot. What is the maximum area of such a plot, given that there are 40 m of fencing available?



SOLUTION. Let y be the length (parallel to the wall) and x the width (perpendicular to the wall) of the rectangular plot, as in the annotated diagram. The area of the plot is then xy and, if we use all of the fencing, we have $2x + y = 40$.

It follows from the last equation that $y = 40 - 2x$, and so the area of the plot in terms of x alone is $A(x) = xy = x(40 - 2x) = 4x - 2x^2$. Note that $0 \leq x$ since x is a length and since y is also a length, we must have $0 \leq y = 40x - 2x$, from which it follows that $2x \leq 40$, *i.e.* $x \leq 20$. We thus need to maximize $A(x) = 40x - 2x^2$ for $0 \leq x \leq 2$. As usual, we find the critical point(s) of $A(x)$ in the interval and compare the value(s) of $A(x)$ at such point(s) with its values at the endpoints of the interval.

$$A'(x) = \frac{d}{dx}\left(40x - 2x^2\right) = 40 - 4x = 4(10 - x) = 0 \iff x = 10$$

The value at the sole critical point, $x = 10$, which happens to be the midpoint of the interval, is $A(10) = 40 \cdot 10 - 2 \cdot 10^2 = 400 - 200 = 200$. The values at the endpoints of the interval, $x = 0$ and $x = 20$, are $A(0) = 40 \cdot 0 - 2 \cdot 0^2 = 0$ and $A(20) = 40 \cdot 20 - 2 \cdot 20^2 = 800 - 2 \cdot 400 = 0$, respectively.

It follows that the maximum area of a plot meeting the given conditions is 200 m^2. $\blacksquare$