Mathematics 1110H (Section B) – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Fall 2023

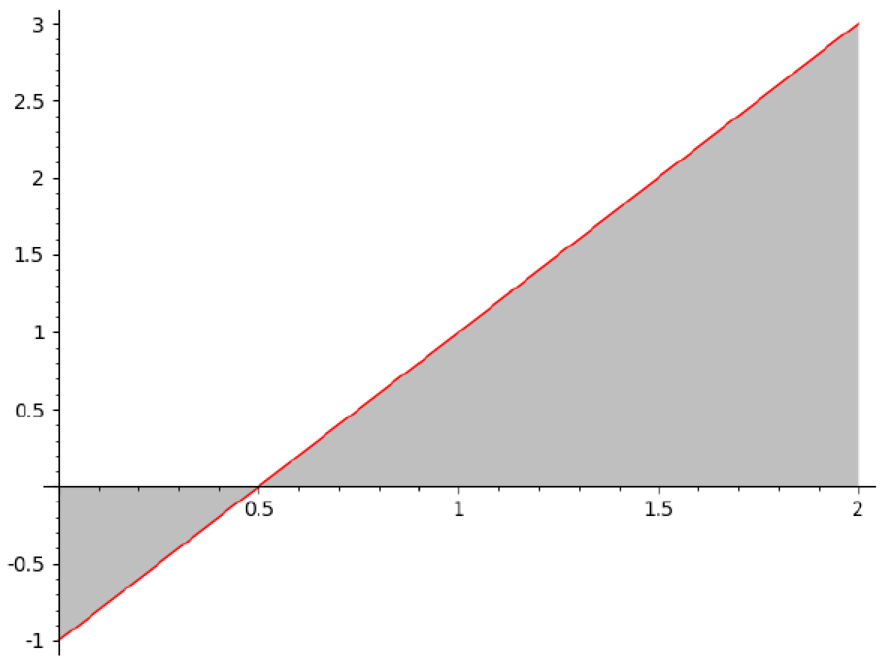
Solutions to Quiz #9 *[Corrected 2023-11-27.]*
Some Definite Integrals

REMINDER. While you are allowed to work together and look things up when doing the quizzes and assignments, your submission should be written up entirely by yourself, giving credit to any collaborators or sources that you ended up actually using. *Please show all your steps and simplify your answers as far as practical.*

1. Without actually doing any calculus, evaluate $\displaystyle\int_0^2 (2x-1)\,dx$. *[1]*

SOLUTION 1. *Understanding.* A definite integral represents the weighted area between the graph of the integrand and the x-axis, where the area above the x-axis is positive and the area below the x-axis is negative. Per the hint after question **2**, the area represented by the integral looks like:

[1]:
```
plot( 2*x - 1, 0, 2, color='red', fill=True )
```
[1]:



The area consists of two triangles, one of width $\frac{1}{2} = 0.5$ and height 1 and hence area $\frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \frac{1}{4} = 0.25$ below the x-axis, and one of width $\frac{3}{2} = 1.5$ and height 3 and hence area $\frac{1}{2} \cdot \frac{3}{2} \cdot 3 = \frac{9}{4} = 2.25$. Since area below the x-axis is negative and area above the x-axis is positive, the weighted area represented by the integral is $\frac{9}{4} - \frac{1}{4} = \frac{8}{4} = 2$. Thus $\displaystyle\int_0^2 (2x-1)\,dx = \frac{7}{4} = 2$. □

Solution 2. *Brutal efficiency.* If we turn the job over to SageMath, we're not doing any calculus ourselves, are we?

```
[2]: integral( 2*x - 1, x, 0, 2 )
```

```
[2]: 2
```

$\square$

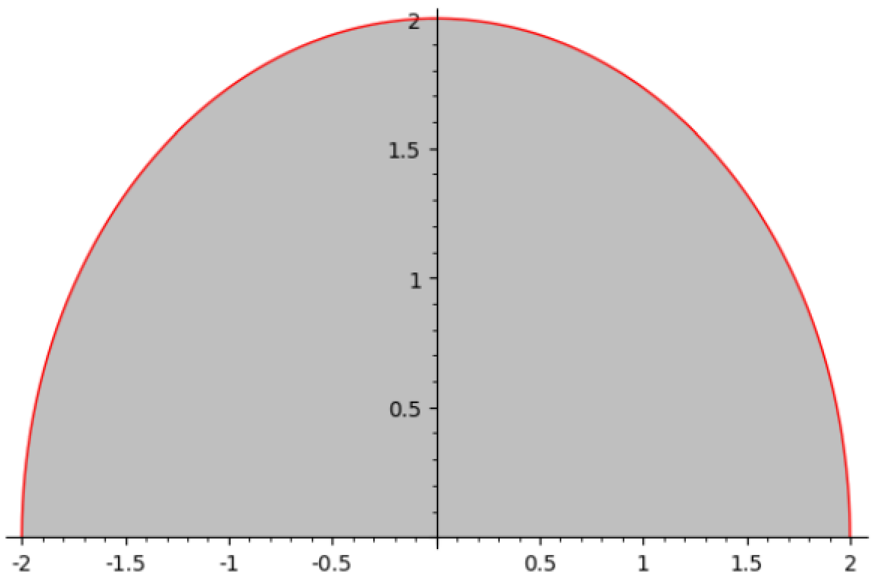
2. Without actually doing any calculus, evaluate $\displaystyle\int_{-2}^{2} \sqrt{4-x^2}\,dx$. *[1]*

Hint: What do the integrals in **1** and **2** represent? Draw the pictures!

Solution 1. Following the hint, we use SageMath again to draw the region in question:

```
[3]: plot( sqrt(4 - x^2), -2, 2, color='red', fill=True )
```

```
[3]:
```



Hmm - allowing for the difference in vertical *vs.* horizontal scaling, this looks like a semi-circle of radius 2. (Check: if $y = \sqrt{4-x^2}$, then $y^2 = 4 - x^2$, so $x^2 + y^2 = 2^2$, which is the equation of a circle of radius 2 centred at the origin.) The area of a semi-circle of radius 2 is half that of the full circle, so it is $\frac{1}{2} \cdot \pi 2^2 = 2\pi$. Since this semi-circle is entirely above the x-axis, the corresponding weighted area is $\displaystyle\int_{-2}^{2} \sqrt{4-x^2}\,dx = 2\pi$. $\square$

Solution 2. *Brutal efficiency strikes again.*

```
[4]: integral( sqrt(4 - x^2), x, -2, 2 )
```

```
[4]: 2*pi
```

$\square$

3. Using calculus, evaluate $\displaystyle\int_0^2 \left(x^2 - 2x - 3\right)\, dx$. *[1]*

Solution. We'll use the fact that integration respects addition, subtraction, and multiplication by constants, as well as the Power Rule:

$$\begin{aligned}\int_0^2 \left(x^2 - 2x - 3\right)\, dx &= \int_0^2 x^2\, dx - 2\int_0^2 x\, dx - 3\int_0^2 1\, dx = \left.\frac{x^3}{3}\right|_0^2 - 2\left.\frac{x^2}{2}\right|_0^2 - 3x|_0^2 \\ &= \left[\frac{2^3}{3} - \frac{0^3}{3}\right] - \left[2^2 - 0^2\right] - \left[3\cdot 2 - 3\cdot 0\right] = \frac{8}{3} - 4 - 6 \\ &= \frac{8}{3} - \frac{12}{3} - \frac{18}{3} = -\frac{22}{3} \quad \square\end{aligned}$$

4. Using calculus, evaluate $\displaystyle\int_{-\pi}^{\pi} \sin(x)\, dx$. *[1]*

Solution. Here we'll use the fact that, since $\dfrac{d}{dx}\cos(x) = -\sin(x)$, the antiderivative of $\sin(x)$ is $-\cos(x)$:

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(x)\, dx &= -\cos(x)|_{-\pi}^{\pi} = (-\cos(\pi)) - (-\cos(-\pi)) \\ &= (-(-1)) - (-(-1)) = 1 - 1 = 0 \quad \square\end{aligned}$$

5. Using calculus, evaluate $\displaystyle\int_{-\pi}^{\pi} |\sin(x)|\, dx$. *[1]*

Hint: Break it up.

Solution. Since $\sin(x) \leq 0$ for $-\pi \leq x \leq 0$ and $\sin(x) \geq 0$ for $0 \leq x \leq \pi$, we have $|\sin(x)| = -\sin(x)$ for $-\pi \leq x \leq 0$ and $|\sin(x)| = \sin(x)$ for $0 \leq x \leq \pi$. We break up and evaluate the given integral accordingly, using the fact that the antiderivative of $\sin(x)$ is $-\cos(x)$:

$$\begin{aligned}\int_{-\pi}^{\pi} |\sin(x)|\, dx &= \int_{-\pi}^{0} (-\sin(x))\, dx + \int_0^{\pi} \sin(x)\, dx \\ &= (-(-\cos(x)))|_{-\pi}^{0} + (-\cos(x))|_0^{\pi} = \cos(x)|_{-\pi}^{0} + (-\cos(x))|_0^{\pi} \\ &= [\cos(0) - \cos(-\pi)] + [(-\cos(\pi)) - (-\cos(0))] \\ &= [1 - (-1)] + [(-(-1)) - (-1)] = 2 + 2 = 4 \quad \blacksquare\end{aligned}$$