Mathematics 1110H (Section B) – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Fall 2023

Solutions to Quiz #2
A Probably Useless Trigonometric Identity

REMINDER. While you are allowed to work together and look things up when doing the quizzes and assignments, your submission should be written up entirely by yourself, giving credit to any collaborators or sources that you ended up actually using.

NOTE. All angles in this course, unless stated otherwise, are measured in radians, as this will be more convenient once we actually start doing calculus. If you are not familiar with radian measure, please read Section 4.1 of the textbook. For this assignment, you might want to note, in particular, that $\frac{\pi}{4}$ radians is the same angle as $45°$.

1. Using the trigonometric identities you know and special values of the basic trigonometric functions, verify that

$$\tan(2x) = \frac{\sin(x)\cos(x)}{\left(\cos(x) + \sin\left(\frac{\pi}{4}\right)\right)\left(\cos(x) - \sin\left(\frac{\pi}{4}\right)\right)}$$

for those values of x for which this equation is defined. *[4]*

SOLUTION. We will rewrite $\tan(2x)$ in terms of $\sin(x)$ and $\cos(x)$ with the help of the double-angle formulas for sin and cos, and then apply some algebra along with making use of the fact that $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. Here goes:

$$\begin{aligned}
\tan(2x) &= \frac{\sin(2x)}{\cos(2x)} = \frac{2\sin(x)\cos(x)}{2\cos^2(x) - 1} = \frac{2\sin(x)\cos(x)}{2\left(\cos^2(x) - \frac{1}{2}\right)} \\
&= \frac{\sin(x)\cos(x)}{\cos^2(x) - \frac{1}{2}} = \frac{\sin(x)\cos(x)}{\cos^2(x) - \left(\frac{1}{\sqrt{2}}\right)^2} = \frac{\sin(x)\cos(x)}{(\cos(x))^2 - \left(\sin\left(\frac{\pi}{4}\right)\right)^2} \\
&= \frac{\sin(x)\cos(x)}{\left(\cos(x) + \sin\left(\frac{\pi}{4}\right)\right)\left(\cos(x) - \sin\left(\frac{\pi}{4}\right)\right)} \qquad \square
\end{aligned}$$

2. For which values of x is the equation in **1** above actually defined? *[1]*

SOLUTION. We consider the left-hand side and right-hand side of the equation separately, just in case there is a difference.

Recall that $\tan(t)$ makes sense for all values of t except when $t = k\pi + \frac{\pi}{2}$ for some integer k, at which points $\tan(t)$ is undefined and its graph has vertical asymptotes. It follows that the left-hand side of the equation, $\tan(2x)$, makes sense for all values of x except when $2x = k\pi + \frac{\pi}{2}$ for some integer k, *i.e.* when $x = k\frac{\pi}{2} + \frac{pi}{4}$ for some integer k.

Since $\sin(x)$ and $\cos(x)$ are defined for all real values of x, the right-hand side of the equation makes sense for all x except when $\left(\cos(x) + \sin\left(\frac{\pi}{4}\right)\right)\left(\cos(x) - \sin\left(\frac{\pi}{4}\right)\right)$, its denominator, is equal to 0. This occurs exactly when $\cos(x) = \pm\sin\left(\frac{\pi}{4}\right) = \pm\frac{1}{\sqrt{2}}$, which happens when $x = k\frac{\pi}{2} + \frac{pi}{4}$ for some integer k, just like for the left-hand side.

Thus the equation makes sense for all $x \neq k\frac{\pi}{2} + \frac{pi}{4}$ for some integer k. ■