Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Fall 2023
Solutions to Assignment #2
Solving equations with SageMath

If you want to get started on this assignment before attending your lab in MATH 1110H, skimming and later referring to as necessary to Sections 1.4.2, 1.8, and 4.8 of Gregory Bard's book *Sage for Undergraduates* (in the SageMath folder in the Course Content section on Blackboard), and perhaps also Appendix G of this book (which is in a separate file from the rest of the book), is probably going to be useful.

1. The Indian mathematician Bhaskara (1114-1185 A.D.), often called Bhaskara II to distinguish him from an earlier mathematician named Bhaskara (*c.* 600-680 A.D.), posed the following problem in a book dedicated to his daughter Lilavati:

> *Seven times half the square root of a flock of geese was observed to march slowly away and two were seen fighting playfully in the water. Say, what was the number of geese?*

a. Restate the problem given above as an equation. *[1]*

SOLUTION. Let x be the number of geese in the flock. We are told that $\frac{7}{2}\sqrt{x}$ of the geese march away and two are left in the water. Presumable, that accounts for all the geese in the flock, which gives us the equation $x = \frac{7}{2}\sqrt{x} + 2$. □

b. Solve the equation you obtained in **a** by hand. Show all your work! *[1]*

SOLUTION. Our strategy will be to eliminate the fraction and the square root and solve the resulting quadratic equation using the quadratic formula. Here goes:

$$x = \frac{7}{2}\sqrt{x} + 2 \implies \frac{7}{2}\sqrt{(x)} = x - 2 \implies 7\sqrt{x} = 2(x-2) = 2x - 4$$

$$\implies 49x = \left(7\sqrt{x}\right)^2 = (2x-4)^2 = 4x^2 - 16x + 16$$

$$\implies 4x^2 - 65x + 16 = 0$$

$$\implies x = \frac{-(-65) \pm \sqrt{(-65)^2 - 4 \cdot 4 \cdot 16}}{2 \cdot 4} = \frac{65 \pm \sqrt{3969}}{8} = \frac{65 \pm 63}{8}$$

$$\implies x = \frac{128}{8} = 16 \quad \text{or} \quad x = \frac{2}{8} = \frac{1}{4}$$

Since a flock of $\frac{1}{4}$ geese makes no sense – nevermind that $x = \frac{1}{4}$ is not a solution of the original equation $x = \frac{7}{2}\sqrt{x} + 2$ – it follows that the flock had 16 geese. □

c. Solve the equation you obtained in **a** using SageMath. [*Hint:* `solve`] *[1]*

SOLUTION. Plugging the equation from **a** into the `solve` command

```
In [1]: solve(7*sqrt(x)/2 + 2 == x, x)

Out[1]: [x == 7/2*sqrt(x) + 2]
```

has SageMath give us a solution that is as lazy as it is useless ... Rearranging the equation a bit before trying again

```
In [2]: solve(-x + 7*sqrt(x)/2 + 2 == 0, x)

Out[2]: [x == 7/2*sqrt(x) + 2]
```

gives us the same useless result. Modifying the equation to get rid of the square root, however,

```
In [3]: solve(49*x == 4*(x-2)^2, x)

Out[3]: [x == 16, x == (1/4)]
```

gives us the same solutions we got in **b**. Success! Of course, we still have to discard that fractional solution ... ∎

NOTE. The moral to the struggle with SageMath above is that we sometimes need to be able to do algebra by hand to put the problem in a form our computer algebra package can take the rest of the way.

2. The *hyperbolic functions* include:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

The names of these function are usually pronounced something like "sinch", "kosh", "tanch", "co-seech", "seech", and "kotch", respectively. They turn out to be closely related to the natural exponential function (obviously) and the trigonometric functions; the latter connections being more obvious when you look at their series expansions and especially when you start looking at them as functions of a complex variable.

   a. Explain why $-1 < \tanh(x) < 1$ for all x. *[1]*

SOLUTION. By definition, $\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, and we also know that $e^t > 0$ for all real numbers t. Since $e^x - e^{-x}$ is a difference of two positive real numbers and $e^x + e^{-x}$ is a sum of the same two positive real numbers, we must have $|e^x - e^{-x}| < |e^x + e^{-x}|$. It follows that $|\tanh(x)| = \left| \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \right| < 1$, and so $-1 < \tanh(x) < 1$. □

   b. Use SageMath to compute $\lim\limits_{x \to -\infty} \tanh(x)$ and $\lim\limits_{x \to \infty} \tanh(x)$. *[Hint:* `lim`*] [1]*

SOLUTION. Here we go:

```
In [4]: lim(tanh(x),x=-oo)

Out[4]: -1
```

```
In [5]: lim(tanh(x),x=oo)

Out[5]: 1
```

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Note the use of two lowercase o's, *i.e.* oo, to denote infinity in SageMath. These limits mean that $y = \tanh(x)$ has horizontal asymptotes of $y = 1$ to the right and $y = -1$ to the left. $\square$

  c. What does the information in **a** and **b** tell you about $y = \tanh(x)$? What does it tell you about the inverse function, $y = \text{arctanh}(x)$, of $\tanh(x)$? *[1]*

SOLUTION. We know from **a** that the entire graph of $y = \tanh(x)$ is between the lines $y = -1$ and $y = 1$, and we know from **b** that $y = \tanh(x)$ approaches $y = -1$ arbitrarily closely as one heads off to the left (*i.e.* as $x \to -\infty$) and approaches $y = 1$ arbitrarily closely as one heads off to the right (*i.e.* as $x \to \infty$). That is, these lines are left and right horizontal asymptotes for $y = \tanh(x)$. Here's a crude sketch; note that $\tanh(0) = 0$:
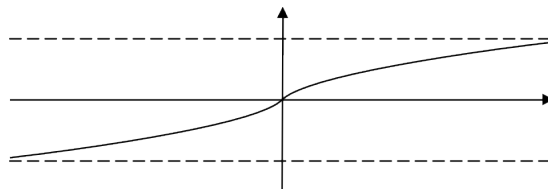
  Since it is the inverse function for tanh, $y = \text{arctanh}(x) \iff x = \tanh(y)$; graphically, this means that the graph of $y = \text{arctanh}(x)$ is the refelction of thew graph of $y = \tanh(x)$ in the line $y = x$. It follows that the entire graph of $\text{arctanh}(x)$ is between the lines $x = -1$ and $x = 1$, and that $y = \text{arctanh}(x) \to -\infty$ as x approaches -1 and $y = \text{arctanh}(x) \to \infty$ as x approaches 1. That is, $x = -1$ and $x = 1$ are vertical asymptotes for $y = \text{arctanh}(x)$. Here's a crude sketch:
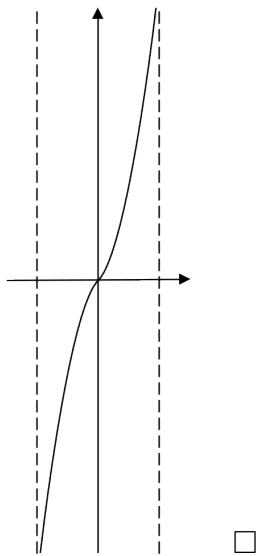
  d. Find a formula for $\text{arctanh}(x)$ by hand. Show all your work. *[1.5]*

SOLUTION. Recall that since they are inverse functions, $y = \text{arctanh}(x) \implies x = \tanh(y)$, so we will try to solve the equation $x = \tanh(y) = \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$ for y in terms of x. Our

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strategy will be to solve for e^y in terms of x first and then get y by applying the natural logarithm function. Here goes:

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}} \implies x\left(e^y + e^{-y}\right) = e^y - e^{-y} \implies x\left(e^y + \frac{1}{e^y}\right)e^y = e^y - \frac{1}{e^y}$$

$$\implies \left(xe^y + \frac{x}{e^y}\right)e^y = \left(e^y - \frac{1}{e^y}\right)e^y \implies x\left(e^y\right)^2 + x = \left(e^y\right)^2 - 1$$

$$\implies x\left(e^y\right)^2 - \left(e^y\right)^2 = -x - 1 \implies (x-1)\left(e^y\right)^2 = -(x+1)$$

$$\implies \left(e^y\right)^2 = \frac{-(x+1)}{x-1} = \frac{1+x}{1-x} \implies e^y = \pm\sqrt{\frac{1+x}{1-x}}$$

Since $e^y > 0$ for all y, we can discard the negative solution.

$$\implies e^y = \sqrt{\frac{1+x}{1-x}} \implies y = \ln\left(\sqrt{\frac{1+x}{1-x}}\right) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Thus the inverse function of $\tanh(x)$ is $\operatorname{arctanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)$.

Note that the various expressions for $\operatorname{arctanh}(x)$ only make sense if both $1 + x$ and $1 - x$ are positive, which requires that $-1 < x < 1$. $\square$

e. Find a formula for $\operatorname{arctanh}(x)$ using SageMath. [*Hint:* `solve`] [1.5]

SOLUTION. We first try the naive approach:

```
In [6]: var("y")
        solve(x==tanh(y),y)

Out[6]: [y == arctanh(x)]
```

Sadly, SageMath knows arctanh natively and just returns that. If we ask it solve for y in terms of x in the equation $x = \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$ we get a better result:

```
In [7]: solve(x==(e^y - e^(-y))/(e^y + e^(-y)), y)

Out[7]: [y == log(-sqrt(-x/(x - 1) - 1/(x - 1))), y == 1/2*log(-x/(x - 1) - 1
        /(x - 1))]
```

It's still up to us to check whether these results make sense; the first one SageMath gives does not because negative numbers are not in the domain of logarithmic functions. Note that SageMath uses log to denote the natural logarithm function we're denoting by ln. $\square$

f. Use SageMath to graph both $y = \tanh(x)$ and $y = \text{arctanh}(x)$ in the same plot, for $-2.5 \le x \le 2.5$ and $-2.5 \le y \le 2.5$. [*Hint:* Add the plots. Literally.] *[1]*

SOLUTION. Here we are, with $y = \tanh(x)$ in green and $y = \text{arctanh}(x)$ in red:

```
In [8]: p1 = plot(tanh(x),-2.5,2.5,color='green')
        p2 = plot(arctanh(x),-2.5,2.5,color='red')
        p1 + p2

        verbose 0 (3899: plot.py, generate_plot_points) WARNING: When plottin
        g, failed to evaluate function at 120 points.
        verbose 0 (3899: plot.py, generate_plot_points) Last error message: '
        Unable to compute f(2.5)'

Out[8]:
```
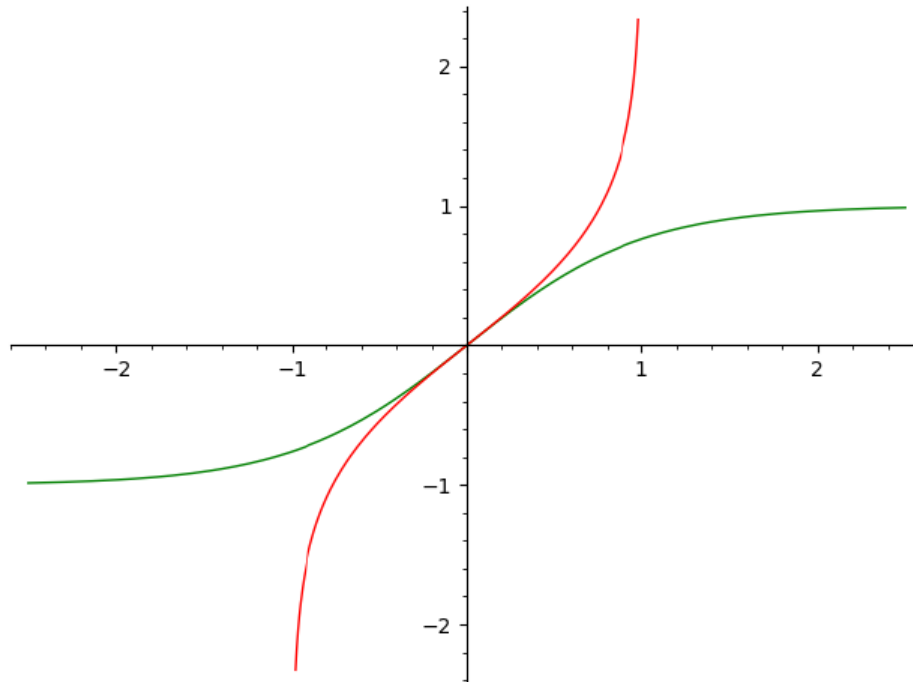

Note that the plots were indeed added together ... ■