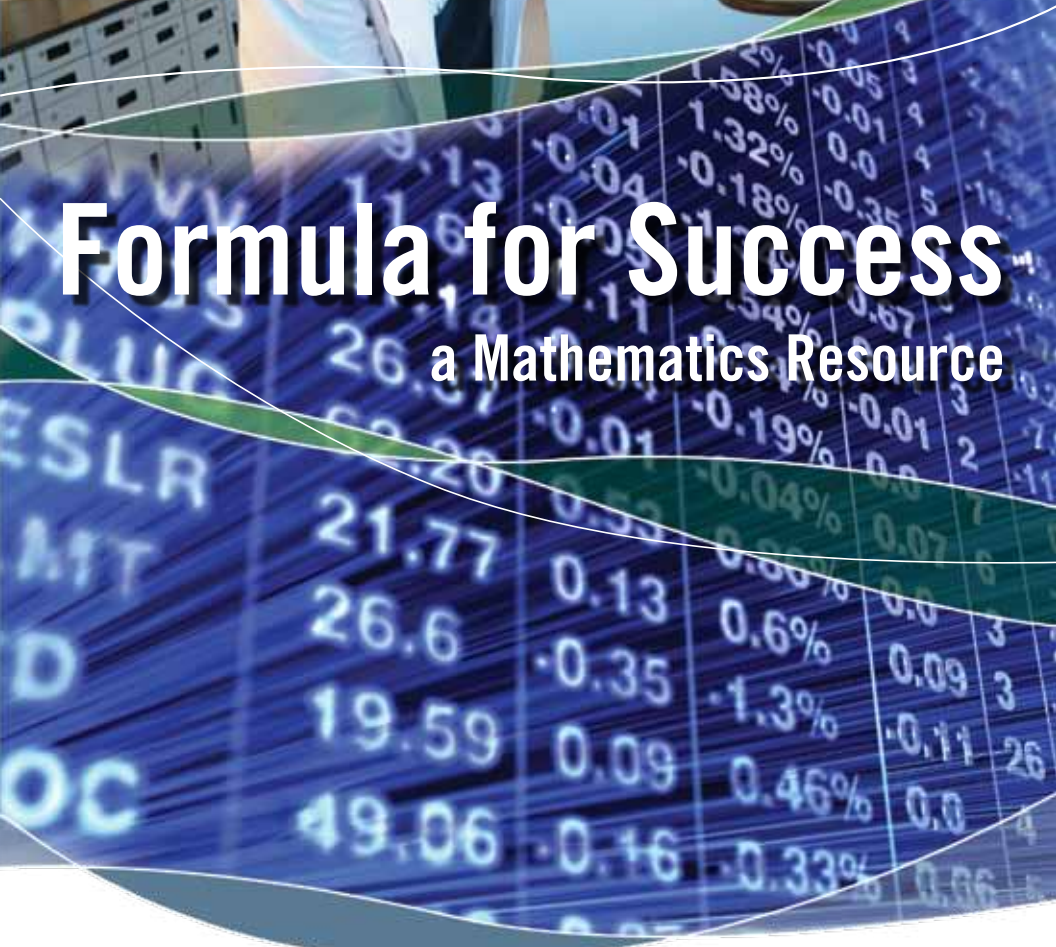


ACADEMIC SKILLS CENTRE (ASC)



Formula for Success

a Mathematics Resource



TRENT UNIVERSITY



Peterborough • Oshawa

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For information on this or any of our services, contact the Academic Skills Centre, Trent University, Peterborough, Ontario K9J 7B8



Phone (705) 748-1720
Fax (705) 748-1830
e-mail: acdskills@trentu.ca
www.trentu.ca/academicskills

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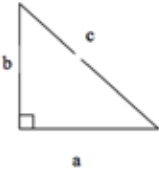
The Academic Skills Centre would like to thank the following for their contributions to this publication:
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Section 1

Formulas and Quick Reference Guide

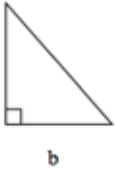
1. Formulas from Geometry

Pythagorean Theorem



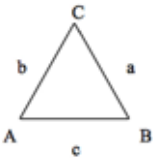
If a triangle has sides a , b , and c (c is the hypotenuse), then $c^2 = a^2 + b^2$

Triangle with Base and Height Given



The area of a triangle whose base is b and altitude is h (the perpendicular distance to the base) is $A = \frac{1}{2} b \times h$

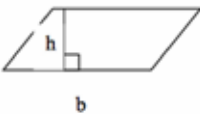
Triangle with Base and Height Unknown



The area of a triangle with angles A , B , and C and sides opposite a , b , and c , respectively:

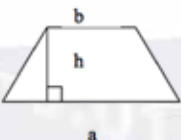
$$\text{Area} = \frac{1}{2} ab \times \sin C$$

Area of a Parallelogram



The area of a parallelogram with base b and altitude h is $\text{Area} = bh$

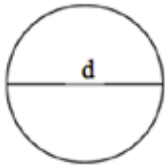
Area of a Trapezoid



The area of a trapezoid whose parallel sides are a and b and altitude is h is

$$\text{Area} = \frac{1}{2}(a + b)h$$

Circle Formulae



For a circle whose radius is r and diameter is d ($d = 2r$):

$$\text{Circumference : } C = 2\pi r = \pi d$$

$$\text{Area: } A = \pi r^2$$

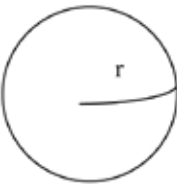


If s is the length of an arc subtended by a central angle of θ radians, then

$$s = r\theta$$

3 Dimensional Solids

Sphere

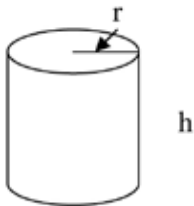


A Sphere with radius r has:

$$\text{Surface Area} = 4\pi r^2$$

$$\text{Volume} = \frac{4}{3} \pi r^3$$

Right Circular Cylinder



A right circular cylinder where r is the radius of the base and h is the altitude has:

$$\text{Surface area} = 2\pi r h$$

$$\text{Volume} = \pi r^2 h$$

Right Circular Cone



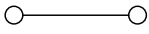
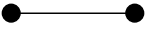


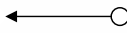
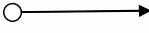



A right circular cone where r is the radius of the base and h is the altitude has:

$$\text{Surface area} = \pi \sqrt{r^2 + h^2}$$

$$\text{Volume} = \frac{1}{3} \pi r^2 h$$

2. Interval Notation

It is assumed that $a < b$

	INTERVAL NOTATION	SET NOTATION	GRAPH
Bounded open interval	(a, b)	$\{x / a < x < b\}$	
Bounded closed interval	$[a, b]$	$\{x a \leq x \leq b\}$	
Bounded half-open interval	$[a, b)$	$\{x a \leq x < b\}$	
Bounded half-open interval	$(a, b]$	$\{x a < x \leq b\}$	
Unbounded open interval	$(-\infty, b)$	$\{x / x < b\}$	
Unbounded open interval	(a, ∞)	$\{x / x > a\}$	
Unbounded closed interval	$(-\infty, b]$	$\{x / x \geq b\}$	
Unbounded closed interval	$[a, \infty)$	$\{x / x \geq a\}$	
Entire real line	$(-\infty, \infty)$	$\{x / x \text{ is a real number}\}$	

3. Co-ordinate Geometry

$$\text{Slope} = m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = -\frac{y_2 - y_1}{x_2 - x_1}$$

Distance between two points A(x_1, y_1) and B (x_2, y_2):

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Distance of any point P (x_1, y_1) from the origin:

$$\sqrt{x_1^2 + y_1^2}$$

Midpoint of the line segment from A (x_1, y_1) to B (x_2, y_2):

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Parallel Lines: $m_1 = m_2$

Perpendicular Lines: $m_2 = -\frac{1}{m_1}$

Slope-point equation of line: $y - y_1 = m(x - x_1)$

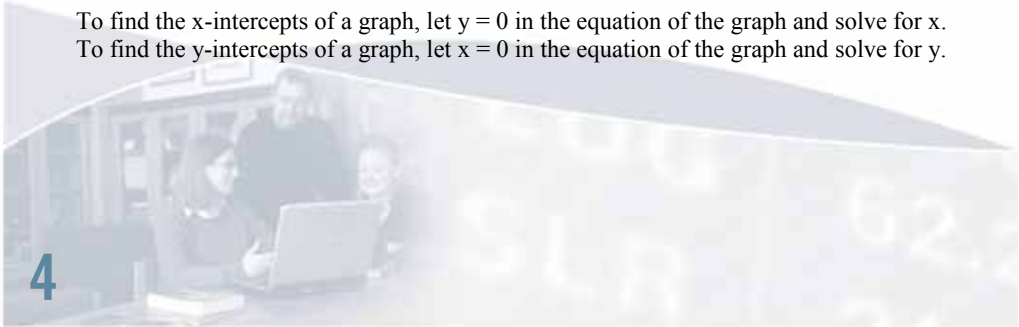
Slope y-intercept equation of line: $y = mx + b$

Line parallel to x-axis: $y = b$ (slope is zero)

Line parallel to y-axis: $x = a$ (slope is undefined)

General equation of a line: $Ax + By + C = 0$

To find the x-intercepts of a graph, let $y = 0$ in the equation of the graph and solve for x.
To find the y-intercepts of a graph, let $x = 0$ in the equation of the graph and solve for y.



4. Steps in Completing the Square

1) Write $ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$

2) Add and subtract the square of half the *new* coefficient of x :

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$$

3) Express the result in the required form $a(x-m)^2 + d$:

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

Example 1: $x^2 + 4x + 5 = x^2 + 4x + (2)^2 - (2)^2 + 5 = (x + 2)^2 + 1$

Example 2:

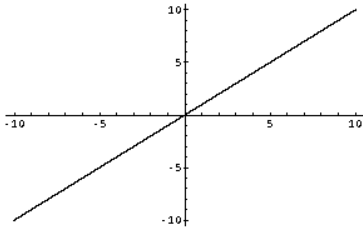
$$3x^2 - 12x + 7 = 3\left(x^2 - 4x + \frac{7}{3}\right) = 3\left[x^2 - 4x + (-2)^2 - (-2)^2 + \frac{7}{3}\right] = 3\left[(x - 2)^2 - \frac{5}{3}\right] = 3(x - 2)^2 - 5$$

Example 3:

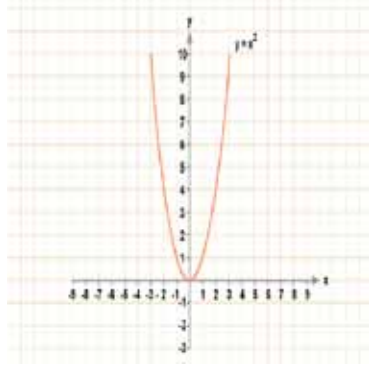
$$-2x^2 + 5x - 4 =$$

$$-2\left(x^2 - \frac{5}{2}x + 2\right) = -2\left[x^2 - \frac{5}{2}x + \left(-\frac{5}{4}\right)^2 - \frac{25}{16} + 2\right] = -2\left[\left(x - \frac{5}{4}\right)^2 + \frac{7}{16}\right] = -2\left(x - \frac{5}{4}\right)^2 - \frac{7}{8}$$

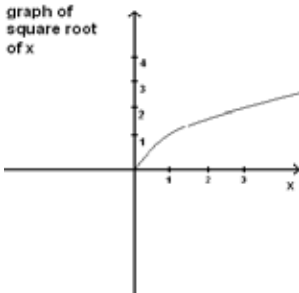
5. The Graphs of the Eight Basic Functions



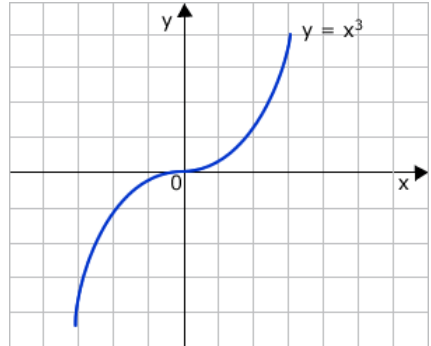
The linear function



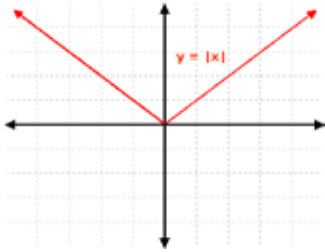
The quadratic function (parabola)



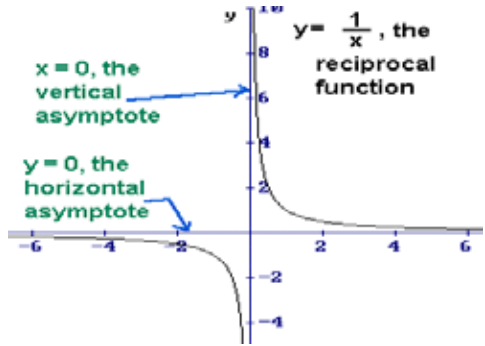
The Square Root Function



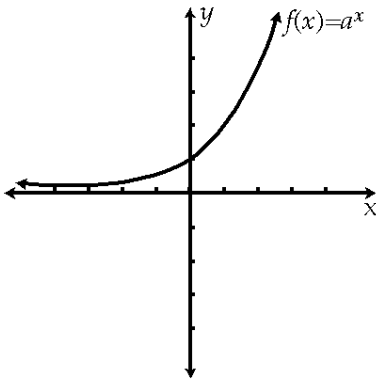
The cubic function



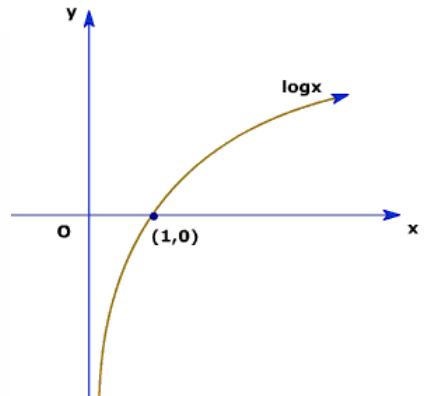
The absolute value function



The reciprocal (rational function). Hyperbola



The exponential function



The logarithmic function



6. Trigonometry - Radian Measure

A circle of radius 1 is known as a unit circle.

The circumference of the unit circle is $2\pi r = 2\pi(1) = 2\pi$.

Since a full revolution through the circle measures 360 degrees, an angle measures 1 degree if it can be obtained by rotating a ray exactly $1/360$ of a complete counter-clockwise revolution.

A positive angle measures 1 radian if, when that angle is placed at the centre of a unit circle, it subtends an arc of length 1.

If a positive angle subtends the entire circumference of a unit circle (when placed at its centre) and since the circumference of a unit circle has length 2π , the radian measure of a full revolution is equal to 2π . Thus, 2π radians = 360 degrees which gives

$$\pi \text{ radians} = 180 \text{ degrees}$$

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees} = 57.3 \text{ degrees}$$

and

$$1 \text{ degree} = \frac{\pi}{180} \text{ radians} = 0.0175 \text{ radian}$$

Example 1: $60^\circ = 60^\circ \times \frac{\pi \text{ radians}}{180^\circ} = \frac{\pi}{3} \text{ radians}$

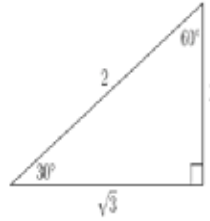
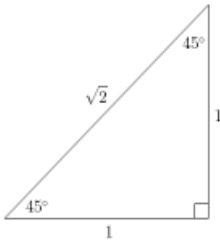
Example 2: $5 \text{ radians} = 5 \text{ radians} \times \frac{180^\circ}{\pi \text{ radians}} = \frac{900}{\pi} \text{ degrees} = 286.5 \text{ degrees}$

A central angle of t radians in a circle of radius r subtends an arc of length $s = rt$. (Remember that the measure of the angle must be in radians, not degrees).



7. Special Angles

Finding The Exact Value of a Trigonometric Ratio



The above triangles, sometimes referred to as “special triangles” enable us to find the exact value of the trigonometric ratios for 30, 45, and 60 degree angles. This is useful when a calculator or table is not available or when it is important to have the EXACT value rather than a rounded off value for the trig ratio.

45, 45, 90 TRIANGLE

This is an isosceles triangle (two angles and two sides equal). If the side length of 1 is assigned to the equal sides, the hypotenuse can be calculated as $\sqrt{1^2 + 1^2} = \sqrt{2}$.

Thus, the exact values are:

$$\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad (\text{rationalized})$$

$$\tan 45^\circ = \frac{1}{1} = 1$$

30, 60, 90 TRIANGLE

This triangle originates from an equilateral triangle with all sides equal to 2 units in length. Each angle is 60°. If you cut the equilateral triangle in half, you get the triangle shown on the right above.

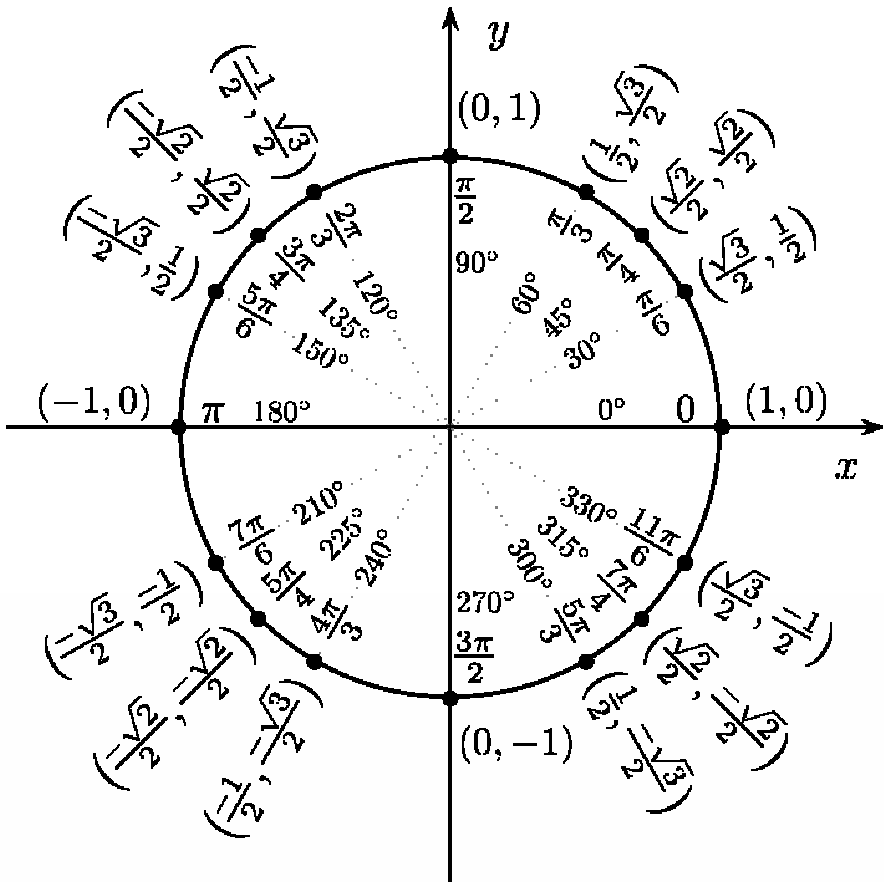
The exact values for the angles in this triangle are:

$$\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$$

$$\sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} \qquad \tan 60^\circ = \frac{\sqrt{3}}{1} = \sqrt{3}$$

8. The Unit Circle



The Unit Circle is one method of finding the exact values for the primary trigonometric functions (without using a calculator or rounding off). If the radius of the circle is 1 unit, the hypotenuse of each right angled triangle formed by moving through the angles 30, 45, 60 and 90° is 1. Starting on the x-axis, the cos and sin of all of the angles shown are written as coordinates of the point formed by going over x units and up y units. The cos is the x coordinate and the sin is the y coordinate. Find tangent by dividing sin by cos.

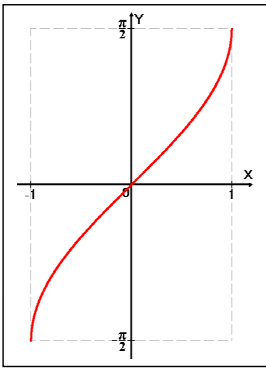
The values in the other 3 quadrants are identical to those in the first quadrant but some will be negative depending on the CAST rule (All positive in Quad 1, Sine positive in Quad 2, Tangent positive in Quad 3, and Cosine positive in Quad 4).

9. Inverse Trigonometric Functions

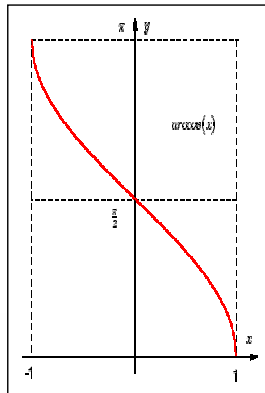
(a.k.a. arcsin, arccos and arctan)

If we restrict the function $f(x) = \sin x$ so that its domain is the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, this function has an inverse. This inverse sine function is denoted by $f^{-1}(x) = \sin^{-1}x$ or arcsin. For example, if $\sin 30^\circ = 0.5$, the inverse function would be to find the angle that has a sine of 0.5, $\sin^{-1}(0.5) = 30^\circ$

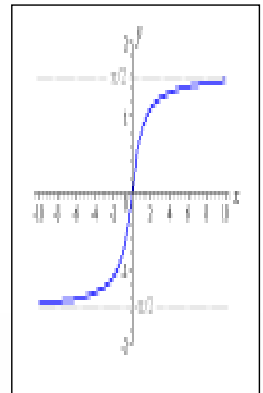
$y = \sin^{-1}x = \arcsin x$ if and only if (iff) $\sin y = x$ for $-1 \leq x \leq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.



$y = \arcsin x$



$y = \arccos x$



$y = \arctan x$

WARNING : the -1 appearing in the notation $f^{-1}(x) = \sin^{-1}x$ is **NOT** an exponent. It denotes the inverse function. It does **NOT** mean $(\sin x)^{-1} = \frac{1}{\sin x}$ (which is the reciprocal of $\sin x$ and is equal to $\csc x$).

The restricted cosine function is the function $g(x) = \cos x$ whose domain is the closed interval $[0, \pi]$. The inverse cosine function is denoted by $g^{-1}(x) = \cos^{-1}x$ or arccos x . Thus, $y = \cos^{-1}x$ iff $\cos y = x$ for $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$.

Examples: $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ since $\cos \frac{\pi}{3} = \frac{1}{2}$

$$\sin^{-1}(-1) = -\frac{\pi}{2} \text{ since } \sin\left(-\frac{\pi}{2}\right) = -1$$

10. Trigonometric Identities

$$P(t) = (\cos t, \sin t)$$

$$\tan t = \frac{\sin t}{\cos t}$$

$$\sec t = \frac{1}{\cos t}$$

$$\csc t = \frac{1}{\sin t}$$

$$\cos t = \frac{\cos t}{\sin t} = \frac{1}{\tan t}$$

PYTHAGOREAN IDENTITIES

$$\sin^2 t = \cos^2 t - 1 \quad 1 + \tan^2 t = \sec^2 t \quad 1 + \cos^2 t = \csc^2 t$$

NEGATIVE ANGLE IDENTITIES

$$\sin(-t) = -\sin t \quad \cos(t) = \cos t \quad \tan(-t) = -\tan t$$

ADDITION AND SUBTRACTION FORMULAS

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A+B) = \frac{\tan B + \tan A}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

DOUBLE ANGLE FORMULAS

$$\sin 2t = 2 \sin t \cos t$$

$$\cos 2t = \cos^2 t - \sin^2 t = 2 \cos^2 t - 1 = 1 - 2 \sin^2 t$$

$$\tan 2t = \frac{2 \tan t}{1 - \tan^2 t}$$

There are many variations of the double angle formulas that can be derived using the above formulas. For example:

$$\cos 6t = \cos^2 3t - \sin^2 3t \quad \sin t = 2 \sin \frac{t}{2} \cos \frac{t}{2}$$

$$\sin^2 t = \frac{1}{2}(1 - \cos 2t) \quad \cos^2 t = \frac{1}{2}(1 + \cos 2t) \quad \text{etc.}$$

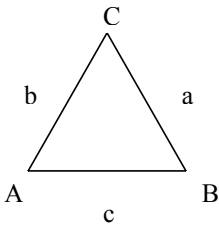
11. Sine Law

A triangle that does not contain a right angle (90°) is called an **oblique** triangle.

The SINE LAW can be used to solve a triangle when:

- i) Two angles and one side are known
- OR
- ii) Two sides and the angle opposite one of them are known.

For triangle ABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$



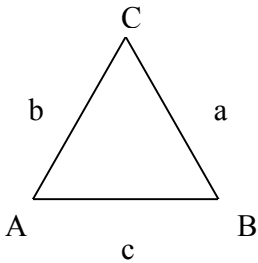
12. Cosine Law

The law of cosines can be used to solve oblique triangles when:

- i) Two sides and the angle between them are known

OR

- ii) All three sides are known.



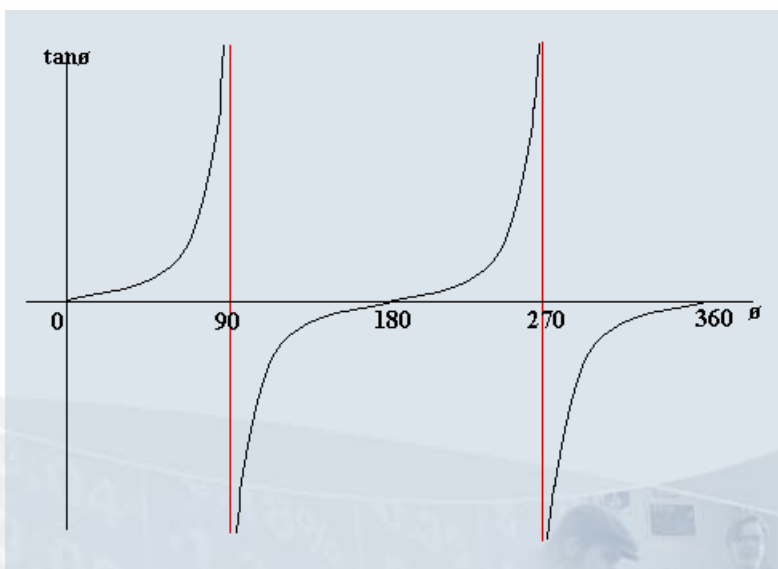
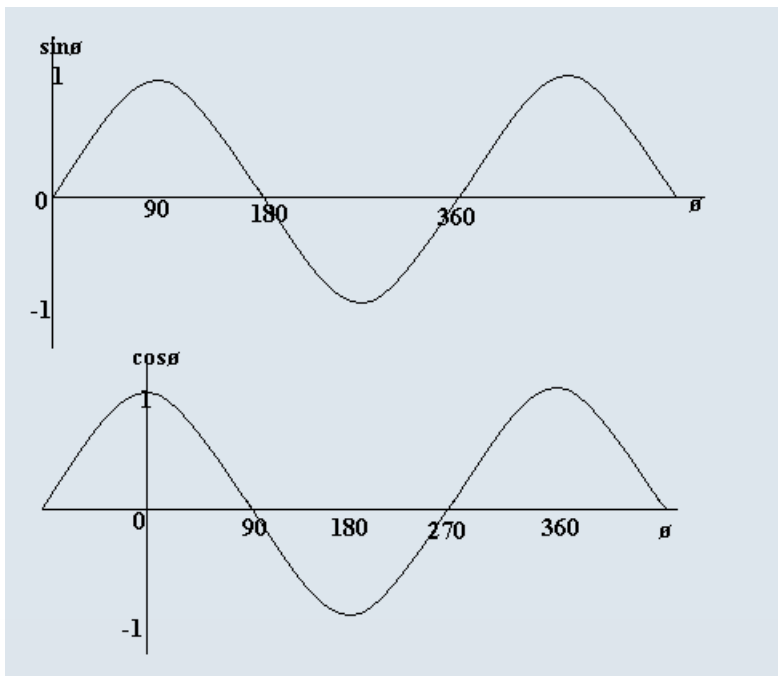
For triangle ABC,

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bccosA \\ b^2 &= a^2 + c^2 - 2accosB \\ c^2 &= a^2 + b^2 - 2abcosC \end{aligned}$$

Also, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$\cos B$ and $\cos C$ can be solved in a similar fashion

13. Graphs of Trigonometric Functions



Section 2

Mathematics Review Modules

1. Order of Operations and Substitution

Order of Operations

The Order of Operations, or BEDMAS, is a very important rule when working with formulae from any math or science topic.

Each letter in the word “**BEDMAS**” stands for a mathematical operation. When an expression involves more than one mathematical operation, they must be done in the correct order to obtain the right answer. Perform multiple operations in the following order:

- B** - compute all expressions inside brackets first. ie. $(3 + 4)$ should be simplified to 7 before performing any other operations. Note that even though the expression $(4)(3)$ uses brackets, they refer in this case to multiplying and should not be treated as brackets in BEDMAS
- E** - the second operation to be performed is to simplify any exponents in the expression ie. $5^3 = 125$.
- D/M** - the next step is to complete any division or multiplication in the question. Do these in The order (left to right) in which they appear in the question.
- A/S** - the last step is to complete any addition or subtraction in the question. Do these in the order in which they appear in the question.

Ex. 1 Evaluate: $10 + 2(18 - 14) - 12 \div 3 + 2$

*Try this problem on your own before comparing it to the answer below.
You might be surprised!!!*

Solution: Following the order of operations (BEDMAS), the correct solution is:

$$\begin{aligned} & 10 + 2(4) - 12 \div 3 + 2 \\ & = 10 + 8 - 12 \div 3 + 2 \\ & = 10 + 8 - 4 + 2 \\ & = 18 - 4 + 2 \\ & = 14 + 2 \\ & = 16 \end{aligned}$$

Common **BEDMAS** errors can result in incorrect results.

For example, the symbol \sum means “sum” (add all the numbers appearing after the sign together). It is important to note that $(\sum x)^2$ means “add all the x values first and then square the final answer” while $\sum x^2$ means “square all of the individual x values first and then add the squares”.

Ex. 2 Find $(\sum x)^2$ and $\sum x^2$ for the following data:

2, 5, 8, 4, 5, 9, 10, 13

Solution: $(\sum x)^2 = (2+5+8+4+5+9+10+13)^2 = 56^2 = 3136$

$$\sum x^2 = 2^2 + 5^2 + 8^2 + 4^2 + 5^2 + 9^2 + 10^2 + 13^2 = 4+25+64+16+25+81+100+169 = 485$$

When working with rational expressions (fractions), the numerator (top) and denominator (bottom) are treated as two separate questions. Apply the rules of BEDMAS to each and solve before performing any operations using both the top and bottom of the fraction.

Ex. 3 Evaluate:

$$\frac{2+7}{2}$$

Solution: Evaluate the top first: $\frac{2+7}{2} = \frac{9}{2} = 4.5$

Note: A common error on this type of problem is to “cross out” the twos on the top and bottom of this expression before evaluating the top. This results in an incorrect answer:

$$\frac{2+7}{2} = 7 \quad \text{As shown above, the correct answer is 4.5, not 7.}$$

Substituting into Formulae

When substituting values into a formula in place of a variable (letter), simply replace the letters with the given number. It is important to note that when two numbers appear together, they must be multiplied. This is the most common source of error in this type of problem.

Remember to always follow the Order of Operations (BEDMAS) when evaluating these expressions.

Ex. 1 If $A = 3$, $X = 4$ and $Y = -5$, Find the value of $B = AX + Y$.

Solution:

$$B = (3)(4) + (-5)$$

$$B = 12 + (-5)$$

$$B = 12 - 5$$

$$B = 7$$

Ex. 2 If $x = 12$ and $y = 6$, Find the value of $\frac{x+y}{x}$

Solution:

$$\frac{x+y}{x}$$

$$= \frac{12+6}{12}$$

$$= \frac{18}{12}$$

$$= 1.5$$

Ex 3. In the formula $A = P(1+i)^n$, Find A if $i = 0.04$, $P = 500$ and $n = 12$

Solution:

$$A = 500(1 + 0.04)^{12}$$

$$= 500(1.04)^{12}$$

$$= 500(1.601)$$

$$= 800.5$$



For Practice Problems on this section, please visit:
www.trentu.ca/academicskills/online_math.php



2. Fractions, Decimals, and Percents

Includes Rounding and Significant Digits

Fractions

Lowest Terms

A fraction is in lowest terms when the numerator (top) and denominator (bottom) have no common factors other than 1 or -1. For example, $\frac{6}{8}$ in lowest terms is $\frac{3}{4}$ (divide the top and bottom by 2).

Mixed Fraction

A mixed fraction is part whole number and part fraction. ie. $1\frac{1}{2}$.

Improper Fraction

An improper fraction is a fraction in which the numerator is larger than the denominator ie. $\frac{13}{5}$

To change an **improper** fraction to a **mixed** fraction:

- divide the numerator by the denominator
- write answer as a whole number
- if there is a remainder, write the remainder as the numerator of a fraction over the existing denominator

Ex 1.: Change $\frac{7}{3}$ to mixed fraction form.

Solution: $7 \div 3 = 2$ with a remainder of 1. $\frac{7}{3} = 2\frac{1}{3}$

To change an **mixed** fraction to an **improper** fraction:

- multiply the whole number by the denominator of the fraction.
- Add this value to the numerator of the fraction.
- Write this answer as the numerator of the fraction using the same denominator.

Ex 2. Change $5\frac{3}{4}$ to improper form.

Solution: Multiply $5 \times 4 = 20$
Add the numerator: $20 + 3 = 23$

The improper fraction is $\frac{23}{4}$

Multiplying Fractions

When multiplying fractions, treat the top and bottom portions of the fractions as separate multiplying questions. Simply multiply the numerators together and write the answer in the numerator and then multiply the denominators and write the answer in the denominator.

$$\frac{P}{Q} \times \frac{R}{S} = \frac{PxR}{QxS} = \frac{PR}{QS}$$

Reduce your answer to lowest terms.

Ex. 3

$$\begin{aligned} & \frac{3}{5} \times \frac{2}{7} \\ &= \frac{6}{35} \end{aligned}$$

Ex. 4

$$\begin{aligned} & \frac{5}{12} \times \frac{2}{3} \\ &= \frac{10}{36} \\ &= \frac{5}{18} \end{aligned}$$

Divide top and bottom by 2 to put the fraction in lowest terms
--

Dividing Fractions

When dividing fractions, a simple change to the question makes it a multiplication question instead. When two fractions are divided, take the reciprocal (turn the fraction upside down) of the second fraction and multiply it by the first according to the rule above.

$$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \times \frac{S}{R} = \frac{PS}{QR}$$

Ex. 5

$$\begin{aligned} & \frac{2}{3} \div \frac{5}{6} \\ &= \frac{2}{3} \times \frac{6}{5} \\ &= \frac{12}{15} = \frac{4}{5} \end{aligned}$$

Adding and Subtracting Fractions

Fractions must have the same denominator before they can be added or subtracted. If the denominator is the same, keep the denominator the same and add the numbers in the numerators

$$\frac{P}{Q} + \frac{R}{Q} = \frac{P+R}{Q}$$

If the two fractions have different denominators, multiply the denominators together to make a new (common) denominator. Multiply each of the numerators by the same number as was multiplied in the respective denominators. Add the fractions as above.

$$\frac{P}{Q} + \frac{R}{S} = \frac{PS}{QS} + \frac{RQ}{QS} = \frac{PS+RQ}{QS}$$

$$\text{Ex 5. } \frac{2}{3} + \frac{5}{7} = \frac{2 \times 7}{3 \times 7} + \frac{5 \times 3}{7 \times 3} = \frac{14+5}{21} = \frac{19}{21}$$

Percents

The per cent sign was used as early as the 1400s. One of the earliest symbols for per cent was the word “per” or just the letter “p” followed by the symbol

p^o

The term percent means “per one hundred”. Thus, when we use percents, we are using numbers as a fraction of one hundred. Over the years, this symbol has evolved. By the mid 1600s, it had become Per

$\frac{o}{o}$

The modern percent sign, % has evolved from this form.

To change a fraction to percent, divide the numerator by the denominator and then multiply the result by 100.

Ex. 1 Change $\frac{4}{5}$ to a percent

Solution: $\frac{4}{5} = 0.8$

$$0.8 \times 100 = 80$$

$$\frac{4}{5} = 80\%$$

To take a given % of a number, multiply the number by the given percent and then divide by 100. (This moves the decimal place two places to the left).

Ex. 2 Find 55% of 200

Solution: $200 \times 55 = 11000$

$$11000/100 = 110$$

$$55\% \text{ of } 200 = 110$$

Decimals

Decimals are another way of writing fractions and percents. When using technology such as calculators and computers, it is easier to work with numbers in their decimal form. There are three different types of decimals,

Terminating eg. 2.25

Repeating eg. $1.\overline{333}$ (the bar over the .333 tells us that the 3's repeat forever)

and

Non-terminating/non-repeating eg. 3.141529..... (this decimal goes on forever without ending or developing a repeating pattern).

When an exact value (ie. Not rounded off) is required, only terminating decimals can be used. Repeating and non-terminating/non-repeating decimals must be rounded off and thus will not give an exact representation of the number.

CONVERTING FROM FRACTIONS TO A DECIMAL

To convert from a fraction to a decimal, divide the numerator (top) of the fraction by the denominator (bottom). If the resulting decimal does not terminate, round off the answer to the required number of decimal places.

Ex. 1 Convert $\frac{13}{20}$ to a decimal

Solution: $13 \div 20 = .65$

Ex. 2 Convert $\frac{3}{11}$ to a decimal. Round your answer to one decimal place.

Solution: $3 \div 11 = \overline{0.2727} \dots$ This is a repeating decimal.

Round off to one decimal place $\Rightarrow = 0.3$ (this is no longer an exact value for the fraction)

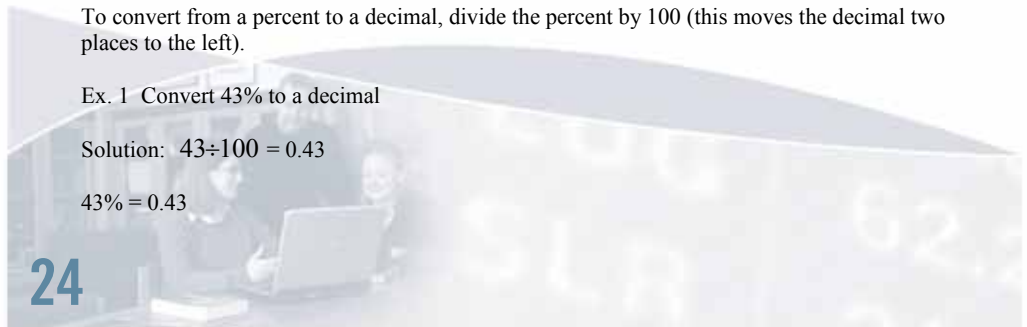
CONVERTING FROM PERCENT TO A DECIMAL

To convert from a percent to a decimal, divide the percent by 100 (this moves the decimal two places to the left).

Ex. 1 Convert 43% to a decimal

Solution: $43 \div 100 = 0.43$

$43\% = 0.43$



Rounding and Significant Digits

ROUNDING

When rounding an answer to a specific number of decimal places, look to the number that is one more decimal place than required.

- If that number is 5 or greater, you will round the previous number up by one.
- If that number is 4 or less, you will leave the previous number as is.

Ex 1. Round 5.167513 to three decimal places

Solution: The number 7 is in the third decimal place. The next number is 5. Since this number is 5 or greater, we will round the 7 up to 8 so the decimal rounded to three decimal places becomes:

5.168

Ex 2. Round 28.3129 to one decimal place

Solution: The number 3 is in the first decimal place. The next number is 1. Since this number is less than 5, we leave the 3 as is so the decimal rounded to one decimal place becomes:

28.3

SIGNIFICANT DIGITS

When rounding numbers, it is also important to consider the appropriate number of significant digits. Significant digits are only those digits that tell us something about the “exactness” of the number.

For example, the number 2000 has only one significant digit (the 2). The zeroes are considered “place holders”. We don’t know if the number 2000 is exactly 2000 or if it has been rounded to the nearest thousand.

The number 2003 has four significant digits. In this case, we know that the number is exact to the ones digit (3).

The number 2000.0 has five significant digits. The decimal tells us that all of the digits to the left of it are significant (exact). The zero on the right of the decimal tells us that this number is exact to the tenths (or first decimal place).

The general rules for significant digits are as follows:

- Any nonzero digit is significant.
- Any zeroes between significant digits are significant.
- Any zeroes which are both to the right of the decimal point and to the right of all non-zero significant digits are significant.

The general rules for rounding to a certain number of decimal places are as follows:

- Do not round off until the end of your calculations (final answer)
- For adding and subtracting, the answer should have the same number of decimal places as the **least number of decimal places** of any of the numbers being added or subtracted.
- For multiplying and dividing, the number of significant digits in an answer should equal the **least number of significant digits** in any one of the numbers being multiplied or divided



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3. Powers, Roots and Factorials

Powers

A power is an expression in the form a^n where “a” is the base and “n” is the exponent. It represents the product of “n” equal factors of “a”.

For example, in the power 3^4 , 3 is the base and 4 is the exponent. This represents a repeated multiplication of the number 3. In this example, the base 3 is multiplied by itself 4 times.

$$3^4 = 3 \times 3 \times 3 \times 3 = 81$$

Smaller powers can be calculated mentally or by hand but larger powers can be calculated using a special button on the scientific calculator. Most calculators have a “ y^x ” or “ x^y ” button for the purpose of calculating powers. To calculate 3^4 , the calculator sequence would be

$$3 \boxed{y^x} 4 = \quad \text{or} \quad 3 \boxed{x^y} 4 = \quad \text{depending on which button your calculator has)}$$

The common name for a power such as 3^4 is “3 to the power of 4”. Powers using the exponents 2 and 3 have special names. Any base (a) to the power of 2 is referred to as “a squared”. This is because whenever you “square” a number, you can always create a square with that number of objects. For example, 4^2 (four squared) equals sixteen. If you arrange 16 objects in a 4×4 pattern as shown, you have created a perfect square.

$$4^2 = 4 \times 4 = 16$$



Similarly, a base, “a” to the exponent 3 is called “a cubed” because a cube can be constructed out of this number of objects. Note that the exponent 2 (squared) creates a 2-dimensional object while the exponent 3 (cubed) creates a 3-dimensional object.

$$2^3 = 2 \times 2 \times 2 = 8$$



TRICKS FOR SQUARING TWO DIGIT NUMBERS

There are several patterns that emerge when squaring larger numbers. This is fun to try but can also be a real time saver if you practice it as you can usually find the answer faster than if you used a calculator.

Here are a couple to get you started:

1) To square any two digit number ending in five (15, 25, 35, etc.), the last two digits of the answer will always be 25. To find the first digit or digits, simply multiply the number in the tens column by one plus that number (ie if the number was a 5 you would multiply by 6, if it was an 8, multiply by 9, etc). Write the answer in front of the 25 and you are done.

For example, 45^2

Any number ending in five that is squared will end in 25. __ _ 25.

For the first digits of the answer, multiply 4 by $(1+4) = 4 \times 5 = 20$

Therefore, $45^2 = 2025$.

2) To square any two digit number, used rounding to get to the nearest ten. For example, for 23^2 we would round to 20. Add the number that you went down from the original number (in this case 3) to the original number ($23 + 3 = 26$). Now multiply 26×20 (just 26×2 with a zero added on the end). $26 \times 2 = 52$. To get the answer to 232, square the amount you rounded down ($3^2=9$) and add this to 560. Therefore, $26^2 = 529$. This trick takes a little more practice but is also very useful, especially when you don't have a calculator nearby!

Eg. Find 31^2

$$31^2 = 30 \times 32 + 1^2 = 960 + 1 = 961$$



Roots

The square root symbol is treated like a bracket in the order of operations. Any operation inside the square root symbol must be done before taking the square root.

Ex. 1 $\sqrt{4+5} = \sqrt{9} = 3$

The square root of a fraction should be treated as the square root of the numerator (top) over the square root of the denominator (bottom).

Ex. 2 $\sqrt{\frac{9}{25}} = \frac{3}{5}$

It is important to note that $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$

Even though $(a)(a) = a^2$ and $(-a)(-a) = a^2$, the symbol \sqrt{a} denotes **only the positive** square root of a .

In the expression $\sqrt{a^2}$, the square root and the exponent 2 are inverses of each other and cancel each other out. The answer to this expression is just “ a ”.

The square root operation can also be expressed using a fractional exponent.

$$\sqrt{a} = a^{\frac{1}{2}}$$

While the square root asks you to find the inverse of a square number, roots such as a “cube root”, “fourth root” etc. can also be found. The symbol $\sqrt[3]{a}$ denotes the “third root” or “cube root” of a . Here we are looking for a number that when multiplied by itself three times gives the answer “ a ”. The cube root can also be written as $a^{\frac{1}{3}}$.

Ex.3 Find $\sqrt[3]{64}$

Since $4^3 = 64$, the cube root of 64 is 4. $\sqrt[3]{64} = 4$

Laws of Exponents

When the bases are the same the following rules apply.

1. Multiplication Law $(x^a)(x^b) = x^{a+b} \Rightarrow 7^8 \cdot 7^4 = 7^{12}$

2. Division Law $x^m \div x^n = x^{m-n} \Rightarrow 3^4 \div 3^7 = 3^{-3}$

3. Power Laws $(x^a)^b = x^{ab} \Rightarrow (4^3)^2 = 4^3 \cdot 4^3 = 4^6$

$$(xy)^a = x^a y^a \Rightarrow (3x^2)^3 = 3^3 x^{2 \cdot 3} = 27x^6$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} = x^a y^{-a} \Rightarrow \left(\frac{4}{5}\right)^2 = \frac{16}{25}$$

4. Zero Exponent $x^0 = 1 \Rightarrow (-425)^0 = 1$

$$\Rightarrow -(425)^0 = -1$$

5. Negative Exponent $x^{-n} = \frac{1}{x^n} \Rightarrow 5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

6. Fractional Exponent $x^{\frac{1}{n}} = \sqrt[n]{x} \Rightarrow 27^{\frac{1}{3}} = \sqrt[3]{27} = 3$

7. Rational Exponent $x^{\frac{a}{b}} = \sqrt[b]{x^a} \Rightarrow 25^{\frac{3}{2}} = \sqrt[2]{25^3} = 5^3 = 125$

$$\Rightarrow 25^{\frac{-3}{2}} = \sqrt[2]{25^{-3}} = \frac{1}{5^3} = \frac{1}{125}$$

Factorials

In mathematics, the **factorial** of a non-negative integer n , denoted by $n!$, is the product of all positive integers less than or equal to n . $n! = n \times (n-1) \times (n-2) \times \dots \times 3 \times 2 \times 1$

For example, $5! = 5 \times 4 \times 3 \times 2 \times 1$

$0!$ has a value of 1

Factorials have a number of applications in probability, statistics, and accounting.

Ex. 1

Find $7!$

Solution: $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

Ex. 2

Find the value of $\frac{8!}{5!}$

Solution:

$$\begin{aligned}\frac{8!}{5!} &= \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} \\ &= 8 \times 7 \times 6 \\ &= 336\end{aligned}$$

Ex. 3

Find the value of $\frac{7!}{2!5!}$

Solution:

$$\begin{aligned}\frac{7!}{2!5!} &= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \cdot 5 \times 4 \times 3 \times 2 \times 1} \\ &= \frac{7 \times 6}{2 \times 1} \\ &= 21\end{aligned}$$



For Practice Problems on this section, please visit:
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Section 3 Calculus/Algebra Prep

1. Essential Algebra

i) Terminology

In order to be able to work with Mathematics courses you will need the terminology that goes along with the work. Some of the terms may be familiar and others will be new. The definitions that follow will be used in many of the Mathematics courses that are offered at university or college.

- absolute value: the distance between any real number and 0 on the number line; for example, $|-3| = 3$ as it is 3 units from 0. Similarly $|+3| = 3$ as it is also 3 units from 0
- algebraic expression: a mathematical expression containing a variable; for example, $2x - 5$ is an algebraic expression
- term: in an algebraic expression each of the parts separated by a + or a - is called a term; for example, in $2x - 5$ there are 2 terms $2x$ and -5 (the + and - signs go with the number they are in front of)
- monomial: an algebraic expression with 1 term; for example, $4a$ is a monomial
- binomial: an algebraic expression with 2 terms; for example $5x^2 - 2$ is a binomial
- trinomial: an algebraic expression with 3 terms; for example, $3x^3 - 2x + 1$ is a trinomial
- polynomial: an algebraic expression with one or more terms (covers all of the above)
- constant: is a term with no variables; in the trinomial above the $+1$ is the constant
- coefficient: the numerical factor of a term; for example, in the term $-6x$ and $-6x^2$, the coefficient is -6
- expanding: multiplying a polynomial by a polynomial
- factoring: writing as a product; to factor a polynomial means to write it as a product of polynomials (the opposite of expanding)
- degree of a term: the sum of the exponents of the variables in that term; for example the degree of $4a$ is 1, the degree of $6a^3b^4$ is 7, the degree of a constant is 0

- degree of a polynomial: the degree of its highest - degree term; for example in $3x^3 - 2x + 1$ the degree is 3
- equation: a mathematical statement that 2 expressions are equal
- inequality: a statement that one quantity is greater than (or less than) another quantity
- linear equation: an equation in which the degree of the highest degree term is 1 for example; $y = 2x - 5$
- quadratic equation: an equation of the form $ax^2 + bx + c = 0$, where a,b,and c are constants and $a \neq 0$
- relation: a rule that produces one or more y values for every valid x coordinate
- function: a relation that gives a single y value for each valid x value
- vertical line test: if no 2 points on a graph can be joined by a vertical line, then the graph represents a function

ii) Expanding and Factoring

Expanding

To expand an expression the process is to multiply it through.

Examples

1. $3a(2a - 7) = 6a^2 - 21a$ where 3a is multiplied through each of the terms in the bracket.

2. $(3a - 5)(2a + 7) = 6a^2 + 21a - 10a - 35$ each term in the first bracket multiplied by each term in the second bracket. This process is sometimes referred to as **FOIL** (first, outside, inside, last).

$= 6a^2 + 11a - 35$ collect terms with like coefficients

Factoring

Factoring is the reverse process to expanding. To factor an expression you must try and put it back into its product form. One of the keys to success is to recognize the kind of factoring that is in the question. Critical to recognition is the number of terms in the expression. Factoring is done to allow you to solve equations and evaluate complicated expressions. **Always look for a common factor in every question that needs factoring.**

Common Factoring - any number of terms

Examples:

- $12x^4y^3 - 8x^3y^2 + 4x^2y$
 $= 4x^2y(3x^2y^2 - 2xy + 1)$
Choose the largest common factor that will divide into each term. Include numbers and letters
- $a(x - y) + b(x - y) = (x - y)(a + b)$
The bracket $(x - y)$ is common to both terms and when it is divided into both $(a + b)$ remains

Often terms can be grouped to find a common factor (even number of terms only)

Examples:

- $ax + bx - ay - by = \underline{ax + bx} - \underline{ay - by}$
 $= x(a + b) - y(a + b)$
 $= (a + b)(x - y)$
group the first 2 and the second 2
common factor each group of terms
take out the common factor
- $ax + 9d + 9a + dx = \underline{ax + 9a} + \underline{dx + 9d}$
 $= a(x + 9) + d(x + 9)$
 $= (x + 9)(a + d)$
group terms 1 and 3; group terms 4 and 2

Difference of Squares - 2 terms both squared

Examples:

1. $b^2 - 25 = (b - 5)(b + 5)$ Take the square root of each of the terms and place in each bracket. Place a + sign in one bracket and a - sign in the other bracket.

2. $81x^4 - 1 = (9x^2 - 1)(9x^2 + 1)$
 $= (3x + 1)(3x - 1)(9x^2 + 1)$

You must re factor when possible.

Trinomial Factoring - 3 terms

A) Easy Trinomial in the form $x^2 + bx + c$ or $x^2 + bxy + cy^2$

Examples

1. $x^2 - 3x - 40 = (x + m)(x + n)$ Find 2 numbers m and n so that $mn = -40$ and $m + n = -3$ since the middle term is created by adding like terms and the last term is created by multiplying the last terms in the brackets. Look at all the factors of -40. In all cases only one pair of factors will add to the required sum.

Here $m = -8$ & $n = 5$ since $-8 + 5 = -3$ and $(-8)(5) = -40$
 $\therefore x^2 - 3x - 40 = (x - 8)(x + 5)$

2. $a^2 + 2ab - 15b^2 = (a + mb)(a + nb)$
 $mn = -15$ and $m + n = 2$

$m = 5$ and $n = -3$ since $5 + (-3) = 2$ & $(5)(-3) = -15$

$\therefore a^2 + 2ab - 15b^2 = (a + 5b)(a - 3b)$ Make sure you multiply out the brackets to **check your answer**

B) Difficult Trinomials in the form $ax^2 + bx + c$ or $ax^2 + bxy + cy^2, a \neq 1, c \neq 1$

Examples

1. $6m^2 + 7m - 20$ Find 2 numbers m and n where $m + n = 7$ (7 is the coefficient of the middle term) and $mn = (6)(-20) = -120$ (120 is the product of the first term coefficient and the constant (last term))
i.e. From all the factors of -120 only $15 + (-8) = 7$
 $\therefore m = 15$ and $n = -8$

Write the middle term as the sum of the 2 terms and then common factor by grouping.

$$\begin{aligned}6m^2 + 7m - 20 &= 6m^2 + 15m - 8m - 20 \\ &= 3m(2m + 5) - 4(2m + 5) \\ &= (2m + 5)(3m - 4)\end{aligned}$$

This method is called **decomposition**.

You can also factor by **trial and error**. You need to find the factors of 6 and of -20 so that the expansion will work. In this case the factors of 6 are 2 and 3 and the factors of -20 are 5 and -4. The 2 and 3 are placed at the beginning of each bracket with the m and the 5 and -4 are at the end of each bracket. With either method you must always **check your result** by expanding.

C) Perfect Square Trinomials

These are trinomials whose answers have 2 brackets that are identical. They may be the easy form or the difficult form.

Examples

1. $w^2 + 12w + 36 = (w + 6)(w + 6)$
 $= (w + 6)^2$
2. $9x^2 - 30x + 25 = (3x - 5)(3x - 5)$
 $= (3x - 5)^2$

Note that the first term in the bracket is the square root ($\sqrt{\quad}$) of the first term in the question and the last term in the bracket is the square root of the last term in the question. The sign in the answer is the middle sign of the question.



Sum or Difference of Cubes - (2 terms) in the form $a^3 + b^3$ or $a^3 - b^3$

Both of these types of factoring follow the same pattern. Follow the pattern to factor similar types of questions. Square of each term in first bracket

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) \quad \text{or} \quad a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Cube root of each term in question;

product of terms in first bracket with opposite sign

Examples

1. $27m^3 - 125n^3$
 $= (3m - 5n)(9m^2 + 15mn + 25n^2)$ The second bracket does not re factor.

2. $\frac{1}{x^3} + 1$

$$= \left(\frac{1}{x} + 1\right)\left(\frac{1}{x^2} - \frac{1}{x} + 1\right)$$

iii) Factor Theorem

(polynomial of degree three or greater)

Reminder: **The Remainder Theorem** allows you to find the remainder when a polynomial is divided by a binomial.

Example

1. $(x^3 + 4x^2 - 2x + 6) \div (x - 3)$

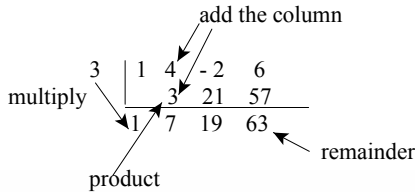
Let $(x - 3) = 0 \therefore x = 3$

Substitute 3 in place of every x in the polynomial to be divided.

ie: $27 + 36 - 6 + 6 = 63$

\therefore the remainder for the division is 63 and the division statement for the question is found when the polynomial is divided.

Synthetic division



$= x^2 + 7x + 19$ remainder 63

Division Statement:

$x^3 + 4x^2 - 2x + 6 = (x - 3)(x^2 + 7x + 19) + 63$

Dividend = (Divisor)(Quotient) + Remainder

2. Find the value of k in $x^3 - 6x^2 + kx - 6$ if the remainder when the polynomial is divided by $(x + 2)$ is 12

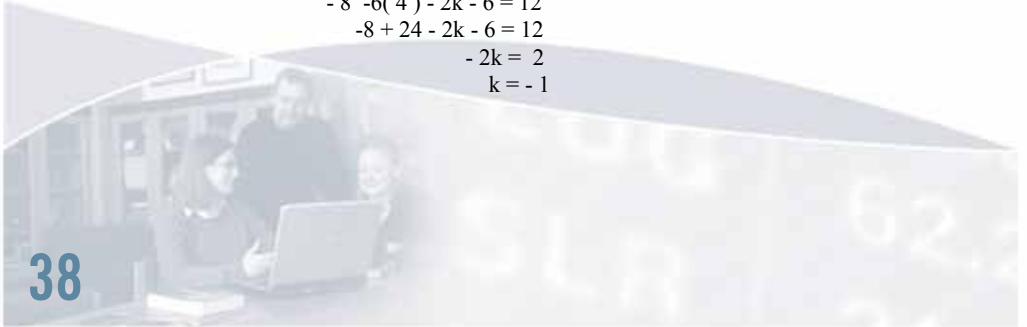
$f(-2) = 12 \therefore (-2)^3 - 6(-2)^2 + k(-2) - 6 = 12$ By substituting -2 for x.

$-8 - 6(4) - 2k - 6 = 12$

$-8 + 24 - 2k - 6 = 12$

$-2k = 2$

$k = -1$



3. Divide $2x^3 - x^2 - 13x - 6$ by $(2x + 1)$

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -1 & -13 & -6 \\ & \underline{-1} & & & \\ & 2 & -2 & -12 & 0 \end{array}$$

Let $2x + 1 = 0$
 $x = -1/2$

Always divide by the denominator of x

$\therefore (2x^3 - x^2 - 13x - 6) \div (2x + 1) = x^2 - x - 6 + 0$ remainder

Always make sure that the dividend is in descending order. Leave spaces for missing terms.
 For example $x^2 - 2x^3 + 1$ rearranges to $-2x^3 + x^2 + 0x + 1$.

If the remainder is zero then the binomial is a factor of the polynomial. Thus using this fact you can choose numbers to substitute into the polynomial to give a zero answer and thus if $f(a) = 0$ then $(x - a)$ is a factor. Then use long division or synthetic division to find the second factor and then re factor if necessary.

Example

1. $x^3 - 4x^2 + x + 6$ Let $f(x) = x^3 - 4x^2 + x + 6$

Now find a number that makes the expression = 0. You must choose numbers that are factors of the constant 6. Choose from $\pm 1, \pm 2, \pm 3, \pm 6$

$f(-1) = -1 - 4 - 1 + 6 = 0$ $\therefore (x + 1)$ is a factor. To find the other factor divide the question by $(x + 1)$

$f(x) = (x + 1)(x^2 - 5x + 6)$

$$\begin{array}{r|rrrr} -1 & 1 & -4 & 1 & 6 \\ & \underline{-1} & & & \\ & 1 & -5 & 6 & 0 \end{array}$$

Re-factor the trinomial. 0 ← remainder

$f(x) = (x + 1)(x - 2)(x - 3)$

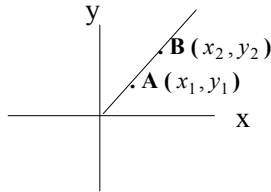


For Practice Problems on this section, please visit:
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2. Equations of Straight Lines

i) Definitions

Any two points on the (x,y) plane when joined form a line segment. If the line segment is extended beyond the two points then it is called a straight line. The points $A(x_1, y_1)$ and $B(x_2, y_2)$ represent the two general points on any line.



The slope “m” of the line containing the points A and B is given by: $m = \frac{y_2 - y_1}{x_2 - x_1}$

The length of the line segment AB or $|AB|$ is given by: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



Equations

For a review of factoring techniques used in this module, please refer to Module One - Algebra

ii) Linear Equations

Examples

1. $4 - 2(x - 3) = 5(1 + 2x)$

$$4 - 2x + 6 = 5 + 10x$$

$$- 12x = - 5$$

$$x = \frac{5}{12}$$

expand and collect like terms

divide both sides by the coefficient of x

You can check your work by substituting the value for x into the left and right sides of the equation. Both sides should balance as they are to be equal to the same amount.

Check L.S. = $4 - 2(\frac{5}{12} - 3)$

$$= 4 - \frac{5}{6} + 6$$

$$= \frac{55}{6}$$

R.S. = $5[1 + 2(\frac{5}{12})]$

$$= 5(1 + \frac{5}{6})$$

$$= 5(\frac{11}{6})$$

$$= \frac{55}{6}$$

∴ the left side = the right side and the value $x = \frac{5}{12}$ is correct. Not each equation needs to be checked on paper but you should always check on your calculator.

2. $\frac{x-3}{2} - \frac{x+4}{3} = 7$

$$6\left(\frac{x-3}{2}\right) - 6\left(\frac{x+4}{3}\right) = 6(7)$$

$$3(x-3) - 2(x+4) = 42$$

$$\downarrow$$
$$x = 59$$

Find the lowest common denominator. LCD = 6

Multiply each term by the LCD.

Continue as in example 1 above and solve for x.

iii) X and Y intercepts

The y - intercept (y_i) is the point where the line crosses the y axis. It is found by letting $x = 0$ in the equation of the line and solving for y . The x - intercept (x_i) is the point where the line crosses the x axis. It is found by letting $y = 0$ and solving for x . Points that are on a line satisfy the equation of the line.

Examples:

1. Determine whether the given points are on the line $3x - 2y + 1 = 0$.

a) $(3, 5)$

$$3x - 2y + 1$$

$$= 3(3) - 2(5) + 1$$

$$= 9 - 10 + 1$$

$$= 0$$

$\therefore (3, 5)$ is on the given line.

Use the left side of the equation.

Substitute in the values for x and y

Evaluate.

Left side = Right side

b) $(-1, 2)$

$$3x - 2y + 1$$

$$= -3 + 4 + 1$$

$$= 2$$

$(-1, 2)$ is not on the line.

Left side \neq Right side

2. Find the x and y intercepts for the line $3x - 2y + 12 = 0$

y_i let $x = 0$

$$-2y + 12 = 0$$

$$y = 6$$

Solve for y .

x_i let $y = 0$

$$3x + 12 = 0$$

$$\therefore x = -4$$

Solve for x .

iv) Quadratic Equations Using Factoring

Examples

1. $x^2 - 5x + 6 = 0$
 $(x - 2)(x - 3) = 0$
 $\therefore x - 2 = 0$ or $x - 3 = 0$
 $\therefore x = 2$ or $x = 3$
- Factor - it's an easy trinomial.
Since these have a product that = 0 each of the factors can = 0.
Solve for each x and check both in the given question.
2. $3x^2 = -6x$
 $3x^2 + 6x = 0$
 $3x(x + 2) = 0$
 $\therefore 3x = 0$ or $x + 2 = 0$
 $\therefore x = 0$ or $x = -2$
- First rearrange so the equation = 0.
Factor - it's a common factor
Set each factor = 0.
Solve for x
3. $(2x+1)(2x-3)+5=(x+8)(x-5)+48$
 $4x^2 - 4x - 3 + 5 = x^2 + 3x - 40 + 48$ Expand
 $3x^2 - 7x - 6 = 0$ Collect like terms and set = 0.
 $(3x + 2)(x - 3) = 0$ Factor - it's a difficult trinomial.
 $\therefore 3x + 2 = 0$ or $x - 3 = 0$ Set equal to 0.
 $\therefore x = \frac{-2}{3}$ or $x = 3$ Solve for x and then check in the equation.

v) Solving Quadratics Using Quadratic Formula

If a quadratic equation will not factor then you must use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It was developed by completing the square of the equation

$$ax^2 + bx + c = 0$$

common factor then take half the x term and square it.

Add and subtract this amount.

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) = -c$$

Factor the first three terms and multiply the last term by a.

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} = -c$$

Collect the constants on the right of the equal sign.

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}$$

Divide both sides by a.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Take the square root of both sides.

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Collect the constants on the right.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula for x solves all quadratic equations where a is the coefficient of x^2 , b is the coefficient of x and c is the constant.

Examples

1. $3x^2 - 2x - 2 = 0$ Does not factor. Use the quadratic formula where $a = 3$, $b = -2$ and $c = -2$

Put in the above values for a, b, and c.

$$x = \frac{2 \pm \sqrt{4 - 4(3)(-2)}}{2(3)} \quad \therefore x = \frac{2 \pm \sqrt{28}}{6} \quad \therefore x = \frac{2 \pm \sqrt{4(7)}}{6} \quad \therefore x = \frac{2 \pm 2\sqrt{7}}{6} \quad \therefore x = \frac{1 \pm \sqrt{7}}{3}$$

There are two values for every quadratic equation. They are called the conjugates of each other. The value under the square root is called the discriminant. This value determines the type of roots (or solutions) to the quadratic equation.

If $b^2 - 4ac > 0$ the equation has two distinct roots (both real numbers)
 If $b^2 - 4ac = 0$ the equation has exactly one root (which is real)
 If $b^2 - 4ac < 0$ the equation has no real roots, but two complex roots

2. $2x^2 - 5x = -8$ Rearrange the equation to equal zero, then apply the quadratic formula.
 Where $a = 2$, $b = -5$, and $c = 8$

$$x = \frac{5 \pm \sqrt{25 - 4(2)(8)}}{4} \quad \therefore x = \frac{5 \pm \sqrt{-39}}{4} \quad \text{Where } i = \sqrt{-1} \quad \therefore x = \frac{5 \pm i\sqrt{39}}{4}$$

The solution to this quadratic equation has 2 complex roots. These are called complex conjugates.

3. What is the conjugate of $7 + 3i$? Since the conjugate has the opposite sign in front of the i term the conjugate is $7 - 3i$. Be careful of order and always change the sign in front of the i term.

4. What is the discriminant of $5x^2 - 8x - 7 = 0$ and what type of roots does it have?

$b^2 - 4ac = 64 - 4(5)(-7) = 204$ Since $204 > 0$ there are 2 distinct real roots.

vi) Absolute Value Equations

An absolute value equation has the variable within the absolute value symbols.

Remember that if x is a real number, then if $x \geq 0$, $|x| = x$

if $x < 0$, $|x| = -x$

Examples

1. $|x| = 2 \Rightarrow$ if $x \geq 0$ then $x = 2$, if $x < 0$ then $x = -2$

Both values of x work in the original equation and are solutions to the equation.

2. $|x + 3| = 7$ You must evaluate both the positive and the negative for the expression inside the absolute.

If $(x + 3) \geq 0$ then $x \geq -3 \Rightarrow x + 3 = 7 \therefore x = 4$ which is larger than -3

If $(x + 3) < 0$ then $x < -3 \Rightarrow -(x + 3) = 7$ or $x + 3 = -7 \therefore x = -10$ which is less than -3

Both answers for x fit their initial condition and are solutions to the equation.

3. $|2x - 3| = 4x - 2$

If $(2x - 3) \geq 0$ then $x \geq \frac{3}{2} \Rightarrow 2x - 3 = 4x - 2 \therefore x = \frac{-1}{2}$ which **does not** meet the initial condition of $x \geq \frac{3}{2}$ and is not a solution to the equation.

If $(2x - 3) < 0$ then $x < \frac{3}{2} \Rightarrow -(2x - 3) = 4x - 2 \Rightarrow -2x + 6 = 4x - 2 \therefore x = \frac{5}{6}$

which **does** meet the initial condition of $x < \frac{3}{2}$ and is the only solution to the equation.



vii) Solving Polynomial Equations (With Degree Three or More)

There are formula for solving cubic and quartic equations and they involve cube and fourth roots. These are cumbersome and difficult to work. Some of these equations can be solved by factoring using the factor theorem or sum and difference of cubes and then using the quadratic formula.

Examples

1. $2x^3 + 10x^2 + 12x = 0$

Common factor first and then factor the quadratic or use the quadratic formula.

$$2x(x^2 + 5x + 6) = 0$$

$$2x(x + 2)(x + 3) = 0$$

Set each factor equal to zero and solve for x.

$$2x = 0 \quad x + 2 = 0 \quad x + 3 = 0$$

$$x = 0 \quad x = -2 \quad x = -3$$

Cubic equations have three solutions. All of them may not be real numbers.

2. $x^3 + 1 = 0$

Factor as the sum of cubes and solve the remaining quadratic using the formula.

$$(x + 1)(x^2 - x + 1) = 0$$

$$x = -1 \text{ or } x = \frac{1 \pm \sqrt{1 - 4(1)(1)}}{2}$$

Simplify the quadratic formula for the other two roots.

$$x = \frac{1 \pm \sqrt{-3}}{2}$$

$$= \frac{1 \pm j\sqrt{3}}{2}$$

There is one real root and two complex roots.

3. $x^3 - 7x + 6 = 0$

Let $f(x) = x^3 - 7x + 6$

This is a cubic and you must use the factor theorem to solve.

$f(1) = 0 \therefore x - 1$ is a factor. Divide using long division or synthetic division.

$$\begin{array}{r|rrrrr} 1 & 1 & 0 & -7 & 6 \\ & \downarrow & & & \\ & 1 & 1 & -6 & \end{array}$$

$f(x) = (x - 1)(x^2 + x - 6)$ Factor the quadratic.

$\therefore f(x) = (x - 1)(x + 3)(x - 2)$

Set $f(x) = 0$ and then set each factor equal to zero.

$\therefore (x - 1)(x + 3)(x - 2) = 0$

$\therefore x = 1, x = -3, x = 2$

4. $x^4 - 5x^2 + 4 = 0$

$(x^2 - 4)(x^2 - 1) = 0$

$(x - 2)(x + 2)(x - 1)(x + 1) = 0$

$\therefore x = \pm 2$ or $x = \pm 1$

This is a trinomial and can be factored as such.

$mn = 4$ and $m + n = -5 \therefore m = -4$ and $n = -1$

Re factor as difference of squares.

5. Write a polynomial equation with roots

a) 2, 5, -1

b) 3, 2 + i, 2 - i

a) Let $x = 2$ $x = 5$ $x = -1$

$\therefore x - 2 = 0, x - 5 = 0, x + 1 = 0$

$\therefore (x - 2)(x - 5)(x + 1) = 0$

$\therefore x^3 - 6x^2 + 3x + 10 = 0$

b) Let $x = 3$ $x = 2 + i$ $x = 2 - i$

$\therefore x - 3 = 0, x - 2 - i = 0, x - 2 + i = 0$

$\therefore (x - 3)(x - 2 - i)(x - 2 + i) = 0$

$\therefore (x - 3)(x^2 - 2x + xi - 2x + 4 - 2i - xi + 2i - i^2) = 0$

$\therefore (x - 3)(x^2 - 4x + 4 - i^2) = 0$ Where $i^2 = -1$

$\therefore (x - 3)[x^2 - 4x + 4 - (-1)] = 0$

$\therefore x^3 - 7x^2 + 17x - 15 = 0$



For Practice Problems on this section, please visit:
www.trentu.ca/academicskills/online_math.php

3. Exponents

i) Laws of Exponents

A power is an expression in the form a^n where “a” is the base and “n” is the exponent. It represents the product of “n” equal factors of “a”. For example $5^6 = (5)(5)(5)(5)(5)(5)$.

Laws of Exponents

When the bases are the same the following rules apply. ($x \neq 0$)

1. Multiplication Law $(x^a)(x^b) = x^{a+b} \Rightarrow 7^8 \cdot 7^4 = 7^{12}$
2. Division Law $x^m \div x^n = x^{m-n} \Rightarrow 3^4 \div 3^7 = 3^{-3}$
3. Power Laws
 $(x^a)^b = x^{ab} \Rightarrow (4^3)^2 = 4^3 \cdot 4^3 = 4^6$
 $(xy)^a = x^a y^a \Rightarrow (3x^2)^3 = 3^3 x^{2 \cdot 3} = 27x^6$
 $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} = x^a y^{-a} \Rightarrow \left(\frac{4}{5}\right)^2 = \frac{16}{25}$
4. Zero Exponent $x^0 = 1 \Rightarrow (-425)^0 = 1$
 $\Rightarrow -(425)^0 = -1$
5. Negative Exponent $x^{-n} = \frac{1}{x^n} \Rightarrow 5^{-2} = \frac{1}{5^2} = \frac{1}{25}$
6. Fractional Exponent $x^{\frac{1}{n}} = \sqrt[n]{x} \Rightarrow 27^{\frac{1}{3}} = \sqrt[3]{27} = 3$
7. Rational Exponent $x^{\frac{a}{b}} = \sqrt[b]{x^a} \Rightarrow 25^{\frac{3}{2}} = \sqrt{25^3} = 5^3 = 125$
 $\Rightarrow 25^{-\frac{3}{2}} = \sqrt[2]{25^{-3}} = \frac{1}{5^3} = \frac{1}{125}$

When doing rational exponents it is usually easier to find the root first and then apply the power. Questions often require positive exponents in the answer so negative exponents are changed into positive exponents by moving the term into the denominator if it was in the numerator or into the numerator if it started in the denominator.

Example : $\frac{6x^{-2}}{y^{-3}z^4} = \frac{6y^3}{x^2z^4}$

Note that only the bases with the negative exponents move. All other bases stay in the same place.

Examples: Use the above laws to evaluate or simplify the following. Leave answers with positive exponents.

$$1. \frac{42a^4}{-3a^{-11}}$$

$$= -14a^{15}$$

Use the Division Law.

Divide the coefficients and subtract the exponents on a.

$$2. (-3m^4)(12m^{-6})$$

$$= -36m^{-2}$$

$$= \frac{-36}{m^2}$$

Use the Multiplication Law.

Multiply the coefficients and add the exponents on m.

Change to positive exponent using $m^{-2} = \frac{1}{m^2}$.

$$3. \frac{(x^{2a})(x^{-5a})}{x^{-3a}}$$

$$= \frac{x^{-3a}}{x^{-3a}}$$

$$= x^0$$

$$= 1$$

Use Multiplication and Division Laws.

Add the exponents in the numerator.

Subtract the exponents.

Any base to a zero exponent equals 1.

$$4. \left(\frac{a^{\frac{1}{2}}}{b^{-2}}\right)^{\frac{2}{3}}$$

$$= \frac{a^{\frac{1}{3}}}{b^{\frac{4}{3}}}$$

$$= a^{\frac{1}{3}}b^{\frac{4}{3}}$$

Use the Power and Division Law.

Multiply the exponents in numerator and denominator by the outside power.

Move the b into the numerator to change to a positive exponent.

$$5. \left(\frac{27}{125}\right)^{-\frac{2}{3}}$$

$$= \left(\frac{125}{27}\right)^{\frac{2}{3}}$$

$$= \left(\sqrt[3]{\frac{125}{27}}\right)^2$$

$$= \frac{25}{9}$$

Use the Rational and Negative Exponent Laws.

Invert the fraction to remove the negative exponent.

Change to root and power form.

Find the cube root and then square that answer.

$$6. (2x^4y^{-5})^{-2}$$

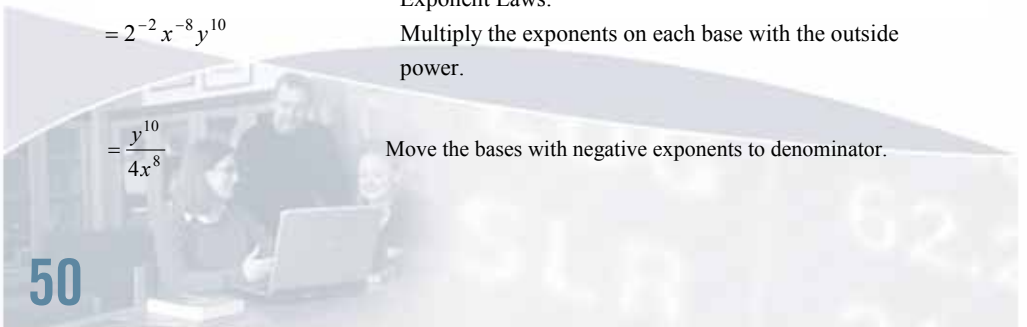
$$= 2^{-2}x^{-8}y^{10}$$

Use the Power and Negative Exponent Laws.

Multiply the exponents on each base with the outside power.

$$= \frac{y^{10}}{4x^8}$$

Move the bases with negative exponents to denominator.



$$\begin{aligned}
 7. \quad & \frac{(2a^2b^4)^3}{(8a^6b^{-12})^{\frac{-2}{3}}} \\
 &= \frac{2^3 a^6 b^{12}}{8^{\frac{-2}{3}} a^{-4} b^8} \\
 &= \frac{8a^{10}b^4}{(\sqrt[3]{8})^{-2}} \\
 &= \frac{8a^{10}b^4}{\frac{1}{4}} \\
 &= 32a^{10}b^4
 \end{aligned}$$

Use the Power and Division Laws.

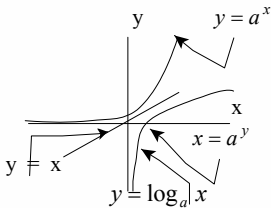
Multiply the exponents in the numerator by 3 and in the denominator by $\frac{-2}{3}$.

Subtract the exponents of a and b and simplify the numeric coefficients.

Evaluate the denominator.

Divide the numerator by the denominator.

ii) The Exponential Function and Its Inverse The Logarithmic Function



The exponential function and its inverse are shown on the sketch to the left. To find the inverse switch the x and y in the equation and reflect the graph through the line $y = x$. This new equation $x = a^y$ is a function as it passes the vertical line test for functions. In order to write this new equation in terms of y requires new terminology. A logarithm is the exponent to which the base “ a ” is raised to produce the answer “ x ”. Logarithm is shortened to “log”.

$x = a^y \Rightarrow y = \log_a x$, $x > 0$, $a > 0$ which is read “ y ” is the exponent to which the base “ a ” is raised to produce the answer “ x ”. Note that both the base and the answer must be positive. When evaluating a logarithm you want to find the exponent for the given base.

Exponent Form	Logarithmic Form
$2^4 = 16$	$\log_2 16 = 4$
$3^2 = 9$	$\log_3 9 = 2$
$9^{\frac{3}{2}} = 27$	$\log_9 27 = \frac{3}{2}$
$x^0 = 1$	$\log_x 1 = 0$
$3^1 = 3$	$\log_3 3 = 1$

Examples:

1. Evaluate each of the following.

a) $\log_3 81$

= 4

Since $3^4 = 81$ and the exponent is the answer to a log.

b) $\log_5 25$

= 2

Since $5^2 = 25$ and the answer is the exponent.

c) $\log_{49} 7$

= .5

Since $49^{\frac{1}{2}} = 7$ and the exponent for 49 that gives 7 is .5.

2. Change to exponent form.

a) $\log_{20} 400 = 2$

$20^2 = 400$

400 is the answer when the base is 20 and the exponent is 2

b) $\log_{\frac{1}{2}} 8 = -3$

$\left(\frac{1}{2}\right)^{-3} = 8$

c) $\log_r s = t$

$r^t = s$

3. Change each of the following to logarithmic form.

a) $6^2 = 36$

$\log_6 36 = 2$

When the base is 6 and the answer is 36 the exponent is 2.

b) $4^{-2} = \frac{1}{16}$

$\log_4 \frac{1}{16} = -2$

c) $x^y = z$

$\log_x z = y$



iii) The Common Logarithm

The common log, the one on the calculator, has a base of 10. The base is most often not written so when the question asked is $\log 100$, the base is understood to be 10. The “log” function on the calculator will find the exponent to which 10 is raised for the input number. Remember that the answer to a “log” question is the exponent. If you input $\log 100$ the answer on the calculator is 2 since $10^2 = 100$.

Examples:

1. $\log 1000$

$$= 3$$

$$\text{Since } 10^3 = 1000$$

2. $\log 450$

$$= 2.65$$

The answer should be between 2 and 3 as 450 is between 10^2 and 10^3 .

3. $\log .1$

$$= -1$$

$$\text{Since } 10^{-1} = \frac{1}{10} = .1.$$

iv) The Natural Logarithm (ln and e)

This logarithm has a base of “e”, Euler’s number and is denoted as “ln” on the calculator. Thus $\log_e x = \ln x$. This is read as “lawn x”. Euler’s number e is used extensively in calculus and is approximately equal to 2.718.

v) Laws of Logarithms

Since logarithms are the exponents they will follow the same rules as exponents. These rules allow you to combine and separate logarithms with the same base.

1. Multiplication Law $\log_a(xy) = \log_a x + \log_a y \Rightarrow \log_5 20 = \log_5 4 + \log_5 5$

2. Division Law $\log_a \frac{x}{y} = \log_a x - \log_a y \Rightarrow \log_2 \left(\frac{3}{4}\right) = \log_2 3 - \log_2 4$

3. Power Law $\log_a x^p = p \log_a x \Rightarrow \log_4 5^2 = 2 \log_4 5$

4. Change of Base Law $\log_a x = \frac{\log_b x}{\log_b a} \Rightarrow \log_2 12 = \frac{\log_{10} 12}{\log_{10} 2}$

5. Exponent Log Law $a^{\log_a x} = x \Rightarrow 2^{\log_2 8} = 8$

Examples: Use the above laws of logarithms to do the following questions.

1. Write as a single logarithm and then evaluate.

a) $\log_3 6 + \log_3 1.5$

$$= \log_3 [(6)(1.5)]$$

$$= \log_3 9$$

$$= 2$$

Use the multiplication law of logs as the bases are the same

Multiply the values in the brackets.

Since the exponent of base 3 that equals 9 is 2.

b) $\log_5 50 - \log_5 4$

$$= \log_5 \left(\frac{50}{4}\right)$$

$$= \log_5 12.5$$

$$= 3$$

Use the division law of logs as the bases are the same.

Divide the values in the brackets.

Since $5^3 = 125$.

c) $2 \log_4 8$

$$= \log_4 8^2$$

$$= \log_4 64$$

$$= 3$$

Use the power law of logs.

Square the value.

Since $4^3 = 64$

2. If $\log_2 7 = 2.81$ use the laws of logs to find the following.

- a) $\log_2 14$ Use multiplication law to split 14 into factors of the base 2 and the value of the answer 7.
 $= \log_2 (2)(7)$ Split the single log into 2 logs that are added.
 $= \log_2 2 + \log_2 7$ Evaluate using the value given for $\log_2 7$ and the fact that $2^1 = 2$.
 $= 1 + 2.81$
 $= 3.81$
- b) $\log_2 49$ Use the power law to evaluate.
 $= \log_2 7^2$ Move the exponent to in front of the logarithm.
 $= 2 \log_2 7$ Use the given value and multiply by 2.
 $= 2(2.81)$
 $= 5.62$
- c) $\log_2 \left(\frac{4}{7}\right)$ Use the division law to split into 2 logs that are subtracted.
 $= \log_2 4 - \log_2 7$ Use the given value for $\log_2 7$ and $4 = 2^2$.
 $= 2 - 2.81$
 $= - .19$

3. Evaluate.

- a) $\log_4 75$ Use the change of base law.
 $= \frac{\log 75}{\log 4}$ Use either base 10 or base e as both are on the calculator.
 ≈ 3.114 Use the calculator to evaluate.
- b) $\log_2 25$ Use the change of base law.
 $= \frac{\ln 25}{\ln 2}$ This time base e was used. Base 10 will give the same answer. Check to make sure.
 ≈ 4.644
- c) $e^{\ln 15}$ Use exponent logarithm law.
 $= 15$

vi) Solving Equations with Exponents or Logarithms

Equations that involve either exponents or logarithms are solved using both the rules of exponents and logs combined with the rules for equation solving. Most often when solving a logarithmic equation it is changed to its exponential form and when solving an exponential equation the logarithm of both sides is taken. When the calculator is used the answer is often approximate.

Examples: Solve for x .

- $\log_3 2 = x$ Use the change of base law.
 $\frac{\log 2}{\log 3} = x$ Use the calculator to solve.
 $x \approx .631$
- $\log_8 (x + 2) - \log_8 (x - 1) = 1$ Use the division law of logs to combine to one logarithm.
 $\log_8 \left(\frac{x+2}{x-1} \right) = 1$ Change to exponential form.
 $8^1 = \frac{x+2}{x-1}$ Multiply both sides by $(x - 1)$.
 $8(x - 1) = x + 2$
 $8x - 8 = x + 2$
 $7x = 10$ Collect like terms and solve for x .
 $x = \frac{10}{7}$
- $2 \log x = \log 8 + \log 2$ Use the multiplication and power laws of logs to have only one logarithm on each side. Equate what is after the logs.
 $\log x^2 = \log 16$ Take the square root of each side.
 $x^2 = 16$
 $x = \pm 4$ Check in the original equation and discard the -4 .
 $x = 4, x \neq -4$
- $\log_2 (x + 1) + \log_2 x = 1$ Use the multiplication law of logs to combine on the left.
 $\log_2 x(x + 1) = 1$ Change to the exponent form.
 $2^1 = x^2 + x$ Collect all terms on one side of the equal sign to $= 0$.
 $x^2 + x - 2 = 0$ Factor the trinomial.
 $(x + 2)(x - 1) = 0$ Solve for x . Check both answers in the original equation.
 $x = -2$ or $x = 1$ $x > 0$ as you can not take the log of a negative number.
 $\therefore x = 1$

5. $2^x = 64$

$$2^x = 2^6$$

$$x = 6$$

Change the right side to a base of 2.
Compare exponents and solve for x.

6. $3^{2x-1} = 30$

$$\log 3^{2x-1} = \log 30$$

$$(2x - 1) \log 3 = \log 30$$

$$2x - 1 = \frac{\log 30}{\log 3}$$

$$2x - 1 = 3.096$$

$$2x = 4.096$$

$$x \approx 2.048$$

Take \log_{10} of both sides of the equation.

Use the power law of logs to put the exponent in front.

Divide both sides by $\log 3$.

Evaluate using the calculator.

Solve for x.

7. $(\log x)^2 - \log x = 20$

$$(\log x)^2 - \log x - 20 = 0$$

$$(\log x - 5)(\log x + 4) = 0$$

$$\log x = 5 \text{ or } \log x = -4$$

$$x = 10^5 \text{ or } x = 10^{-4}$$

Rearrange and treat as a trinomial.

Factor.

Solve each factor.

Rewrite each equation in exponential form.

8. $3^{2x} - 3^x - 6 = 0$

$$(3^x)^2 - 3^x - 6 = 0$$

$$(3^x - 3)(3^x + 2) = 0$$

$$3^x = 3 \text{ or } 3^x = -2$$

$$x = 1 \text{ or } x \log 3 = \log(-2)$$

$$\therefore x = 1$$

Another trinomial.

Factor.

Solve each factor.

Solve for x. There is no solution for the second equation as you can not take the logarithm of a negative number.

vii) Exponential Growth and Decay

In situations where a quantity grows or decays at an exponential rate (ie. compound interest, bacterial growth, radioactive decay) the general equation to determine the final amount is given

by: $y = y_0(1 + i)^{\frac{t}{n}}$ where

- y is the final amount
- y_0 is the initial amount
- i is the percent increase or decrease (written in decimal form)
- t is the time needed to reach the final amount
- n is the time needed for one increase or decrease

Examples:

1. If \$2000 is invested at 7% p.a. compounded semi annually how long will it take for the investment to double? (p.a. means per annum or yearly)

$$y = y_0(1 + i)^{\frac{t}{n}}$$

$$y_0 = \$2000$$

$$y = \$4000 \text{ (double \$2000)}$$

$$i = .07 \div 2 = .035$$

$$n = .5 \text{ (half a year for semi annual)}$$

$$t = ?$$

$$4000 = 2000 (1 + .035)^{\frac{t}{.5}}$$

Substitute all known values into formula.

$$2 = (1 + .035)^{\frac{t}{.5}}$$

Divide both sides by 2000.

$$\log 2 = \frac{t}{.5} \log(1.035)$$

Take the log of both sides and use the power law for logs.

$$\frac{.5 \log 2}{\log 1.035} = t$$

Isolate the "t" value

$$t \approx 10.07 \text{ years}$$

Use the calculator to solve for t.

- ∴ It will take approximately 10.1 years to double the investment at that interest rate.



2. An air filter loses about .3% of its effectiveness each day. What is its effectiveness after 145 days as a percent of its initial effectiveness.

$$y = y_0(1 + i)^n$$

$$y = ?$$

$$y_0 = 100 \text{ (for 100\% effectiveness at the beginning)}$$

$$i = -.003 \text{ (negative as it is losing effectiveness)}$$

$$t = 145$$

$$n = 1 \text{ (for each day)}$$

$$y = 100(1 - .003)^{145}$$

$$y \approx 64.7$$

∴ It will be 64.7% as effective after 145 days.

Substitute all given values into the equation.

Use the calculator to evaluate.

b) If the filter should be replaced when its effectiveness has decreased by 80% after how many days should it be replaced?

$$y = y_0(1 + i)^n$$

$$y = 20 \text{ (100\% - 80\% as it has been reduced by 80\%)}$$

$$y_0 = 100$$

$$i = -.003$$

$$t = ?$$

$$n = 1$$

$$20 = 100(1 - .003)^t$$

$$.2 = .977^t$$

$$\log .2 = t \log .977$$

$$t = \frac{\log .2}{\log .977}$$

$$t \approx 535.7$$

Substitute all given values into the general equation.

Divide both sides by 100.

Take the log of both sides and use the power law for logs.

Solve for t.

∴ It will take 535.7 days before the filter should be replaced.



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4. Functions and Transformations

i) Definition of a Function

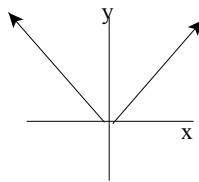
In mathematics a relation between variables indicates how they interact with one another. A “function” is a relation where there is only one output value for each input value. The input variable (usually x) is the independent variable and the output variable (usually y) is the dependent variable. No two ordered pairs (x,y) have the same x coordinate. A function can be represented in several ways. Notice that there is a unique x value for each of the ordered pairs in the examples.

table of values

x	y
0	0
1	1
2	2
-1	1
-2	2

All entries in the x column are different.

graph



No two points can be joined by a vertical line.

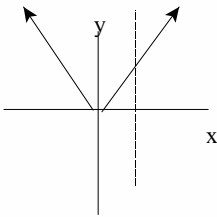
equation

$$y = |x|$$

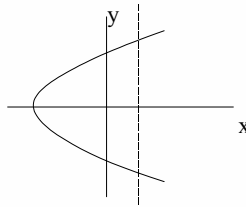
For all values of x there is only one value for y .

Vertical Line Test for a Function

If no two points on a graph can be joined by a vertical line then the graph is a function. The vertical line will only cut the graph once if it is a function.



function
vertical line cuts once



not a function
vertical line cuts more than once



ii) Function Notation

Functions are represented by their own notation appropriately called Function Notation. For each function the y in the equation is replaced by $f(x)$ which is read as “f of x” or “f at x”.

For example the equation $y = 5x - 2$ is a function as it represents a line and so can be written as $f(x) = 5x - 2$. This notation simplifies the recording of values for the function. For example when the value of the function is needed for a particular x value it is recorded as follows: $f(3) = 5(3) - 2$ or $f(3) = 13$. You substitute 3 in place of every x in the function equation to find the y value. Thus one point on the given function is $(3, 13)$.

Algebraic expressions can be substituted in place of the variable as well as numbers.

Examples:

1. Given $g(x) = \frac{x-3}{x}$, $x \neq 0$ Find:

a) $g(2x) = \frac{2x-3}{2x}$

Insert $2x$ in place of each x in the original function.

b) $g(1-x) = \frac{(1-x)-3}{(1-x)}$

Replace x with $(1-x)$.

$$= \frac{-x-2}{1-x}, x \neq 1$$

Simplify by collecting like terms.

2. If $f(x) = 3x - 5$ find

a) $f(1) = 3(1) - 5$
 $= -2$

Substitute 1 for all x in the question.
Evaluate.

b) x when $f(x) = 7$
 $\therefore 3x - 5 = 7$
 $3x = 12$
 $x = 4$

Set $f(x)$ equal to the value
Solve the equation for the value of x .

In a) you are given the x value and are solving for the y or $f(x)$ value and the ordered pair is $(1, -2)$. In b) you are given the y value and are solving for the x coordinate and the ordered pair is $(4, 7)$. Both points are on the graph of $f(x) = 3x - 5$.



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5. Inequalities

i) Inequalities Definition

Inequality: a statement that one quantity is greater than (or less than) another quantity. The inequality may also contain an equal sign for greater than and equal to (or less than and equal to). The symbols used are $>$ (greater than), $<$ (less than). When an equals sign is added below they become \geq (greater than or equal to) or \leq (less than or equal to).

Consider the statement: $6 > 4$

Using the rules for inequalities, and the original statement each time

add 2 to both sides	$8 > 6$] All of these statements are true.
subtract 2 from both sides	$4 > 2$	
multiply both sides by 2	$12 > 8$	
divide both sides by 2	$3 > 2$	

multiply both sides by - 2	$- 12 > - 8$] Both of these statements are false.
divide both sides by - 2	$- 3 > - 2$	

To make the last two statements true the inequality sign must be changed.

Rule: When an inequality is multiplied or divided by a negative number the inequality sign must be reversed.

ii) Linear Inequalities

All the rules for linear inequalities are the same as those for linear equations with the exceptions noted at the beginning. When multiplying or dividing by a negative the inequality sign must be reversed.

Examples

1. $2(x - 3) \leq 5x + 6$
 $2x - 6 \leq 5x + 6$
 $-3x \leq 9$
 $x \geq -3$

Note that you are going to divide by - 3 and the inequality sign reverses in this line.

Solution is the interval $[- 3, \infty)$.

2. $2 < x - 7 < 5$
 $9 < x < 12$

This is a double inequality. Add 7 to all parts of the inequality. The solution is the interval (9, 12). This is all values between 9 and 12 but not including either 9 or 12.

ii) Polynomial Inequalities

(Quadratic, Cubic and Quartic Inequalities)

The following examples are of polynomials that can be factored. Follow all the techniques for polynomial equations to determine the zeros of the equation and then test numbers in those intervals formed by the zeros. The higher the power the more intervals are possible.

Examples

$$1. \quad x^2 - x > 12$$

$$x^2 - x - 12 > 0$$

$$(x - 4)(x + 3) > 0$$

$$\text{Let } (x - 4)(x + 3) = 0$$

$$\begin{array}{c} + \quad | \quad - \quad | \quad + \\ -3 \quad 4 \end{array}$$

$$\therefore x < -3 \text{ or } x > 4$$

$$(-\infty, -3) \text{ or } (4, \infty)$$

Subtract 12 from both sides so that the right is zero.

Factor as an easy trinomial.

The zeros of the equation are 4 and -3.

The intervals are $(-\infty, -3)$, $(-3, 4)$ or $(4, \infty)$. Choose numbers in these intervals and test in the original inequality. Remember you are looking for the ones > 0 .

These are the values of x where the polynomial > 0 .

These are the intervals where the polynomial > 0 .

$$2. \quad 3x(x - 4) \leq 2(x + 1) + 3 \quad \text{Multiply out the brackets and collect all terms on the left.}$$

$$3x^2 - 12x \leq 2x + 2 + 3$$

$$3x^2 - 14x - 5 \leq 0$$

$$(3x + 1)(x - 5) \leq 0$$

$$\text{Let } (3x + 1)(x - 5) = 0$$

$$\begin{array}{c} + \quad | \quad - \quad | \quad + \\ \frac{-1}{3} \quad 5 \end{array}$$

$$\therefore \frac{-1}{3} \leq x \leq 5 \text{ or the interval } \left[\frac{-1}{3}, 5\right] \text{ which is the closed and bounded interval between } \frac{-1}{3} \text{ and } 5 \text{ (this includes both ends of the interval as well).}$$

Factor the resulting trinomial. It is a difficult one.

Set the equation equal to zero to solve for the roots of the polynomial.

The zeros are $\frac{-1}{3}$ or 5. The intervals are $(-\infty, \frac{-1}{3}]$, $[\frac{-1}{3}, 5]$, $[5, \infty)$. Choose where the polynomial ≤ 0 .



$$3. (x-1)(x+3)(x-4) \geq 0$$

$$\text{Let } (x-1)(x+3)(x-4) = 0$$

$$\Rightarrow x = 1, -3, \text{ or } 4$$

$$\begin{array}{ccccccc} - & | & + & | & - & | & + \\ -3 & & 1 & & 4 & & \end{array}$$

$\therefore -3 \leq x \leq 1$ or $x \geq 4$ or the intervals $[-3,1]$ or $[4, \infty)$.

This is already in factored form so set the polynomial = 0

Solve for the zeros of the equation.

Test numbers in the intervals created.

$(-\infty, -3], [-3, 1], [1, 4], [4, \infty)$

Choose the intervals where the polynomial ≥ 0 .

$$4. x^3 - x^2 < 0$$

$$x^2(x-1) < 0$$

$$\text{Let } x^2(x-1) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 1$$

$$\begin{array}{ccccccc} - & | & - & | & + \\ 0 & & 1 & & \end{array}$$

$\therefore x < 1, x \neq 0$ or the intervals $(0, 1)$ or $(-\infty, 0)$

Common factor and then set the polynomial = 0.

Solve for the zeros of the equation.

Test in the intervals $(-\infty, 0), (0, 1)$ or $(1, \infty)$.

Choose the intervals where the polynomial is < 0 .

$$5. x^4 - 10x^2 + 9 > 0$$

$$(x^2 - 1)(x^2 - 9) > 0$$

$$(x-1)(x+1)(x-3)(x+3) > 0$$

$$\text{Let } (x-1)(x+1)(x-3)(x+3) = 0$$

$$\Rightarrow x = \pm 3 \text{ or } x = \pm 1$$

$$\begin{array}{ccccccc} + & | & - & | & + & | & - \\ -3 & & -1 & & 1 & & 3 \end{array}$$

$\therefore x < -3$ or $-1 < x < 1$ or $x > 3$ or the intervals $(-\infty, -3), (-1, 1), (3, \infty)$

Factor as an easy trinomial and then re factor as a difference of squares.

Solve for the zeros of the equation.

Test in the intervals created. $(-\infty, -3), (-3, -1), (-1, 1),$

$(1, 3)$ or $(3, \infty)$

Choose the ones where the polynomial is > 0 .



Polynomial Inequalities

(With Non Real Roots)

As you know some equations have a combination of real and complex roots. Inequalities will also follow the same pattern.

Example

$$1. (x^2 - 16)(x^2 + 3x + 4) \leq 0$$

$$(x - 4)(x + 4)(x^2 + 2x + 4) \leq 0$$

$$\text{Let } (x + 4)(x - 4) = 0$$

$$\Rightarrow x = \pm 4$$

$$\begin{array}{c} + \quad | \quad - \quad | \quad + \\ \hline -4 \quad 4 \end{array}$$

$$\therefore -4 \leq x \leq 4 \quad \text{or the interval } [-4, 4]$$

Factor the first bracket but the second bracket has only non real roots as

$$b^2 - 4ac = 4 - 4(1)(4)$$

$$= 4 - 16 < 0$$

Solve for the zeros of the equation.

Test in the only real intervals $(-\infty, -4]$, $[-4, 4]$, $[4, \infty)$

Choose the ones where the polynomial is ≤ 0

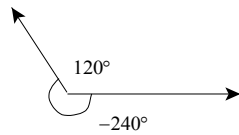


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6. Trigonometric Functions

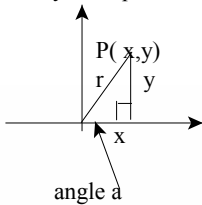
i) Definitions

In trigonometry an angle is represented by a rotation about a point. The amount of rotation is the measure of the angle. An angle is in standard position when the initial arm of the angle is on the positive x axis and the vertex is at the origin. The position of the terminal arm gives the measure of the angle. Counterclockwise rotation gives a positive angle and a clockwise rotation gives a negative angle. This allows for angles that are less than 0° and greater than 180° .



These angles have the same terminal arm and are called coterminal

There are an unlimited angles that are coterminal. They form a complete rotation about the origin of 360° . If the terminal arm of any angle intersects the circle centre $(0,0)$ and radius r given by the equation $x^2 + y^2 = r^2$ then the point $P(x,y)$ is on the circle.



The primary trigonometric ratios created by the angle at the origin between the x axis and the terminal arm are the following:

$$\text{Sine } a = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r} \quad \text{Cosine } a = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r} \quad \text{Tangent } a = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$$



The short form for the ratios is $\sin a$, $\cos a$, and $\tan a$. The reciprocal ratios of each of the primary ratios are as follows:

The reciprocal of $\sin a$ is cosecant a . The reciprocal of $\cos a$ is secant a . The reciprocal of $\tan a$ is cotangent a . The short forms for these ratios are $\csc a$, $\sec a$, and $\cot a$.

In short form the ratios are as follows: (where a represents any angle)

$$\sin a = \frac{y}{r} \quad \csc a = \frac{r}{y}$$

$$\cos a = \frac{x}{r} \quad \sec a = \frac{r}{x}$$

$$\tan a = \frac{y}{x} \quad \cot a = \frac{x}{y}$$

If the circle has a radius of 1 unit (the Unit Circle) then $r = 1$ in the above equations.

The ratios become

$$\sin a = y \quad \text{or} \quad y = \sin a$$

$$\cos a = x \quad \text{or} \quad x = \cos a$$

$$\tan a = \frac{y}{x} \quad \text{or} \quad \tan a = \frac{\sin a}{\cos a}$$

Therefore any point on the unit circle has coordinates $P(\cos a, \sin a)$. As the terminal arm rotates counterclockwise about the origin through all the angles in the 1st quadrant all the way around to the 4th quadrant all angles from 0° to 360° are covered. Angles larger than 360° are found by continuing to rotate about the origin. Angles of less than 0° are found by rotating clockwise. The values for $\sin a$ and $\cos a$ change as the values for x and y change. They are positive or negative depending on the quadrant but never exceed 1 or go below -1. This allows us to determine the sign of the ratio by knowing which quadrant it lies in.

ii) CAST Rule

The CAST rule is a memory aid to remember which ratios are positive in each of the 4 quadrants. In the 1st quadrant values for P(x,y) are positive and so all the ratios are positive. In the 2nd the values for x are negative and y are positive so cos a is negative while sin a is positive and tan a is negative. If you continue through all 4 quadrants you will have a chart like the one below.

sin a + cos a - tan a -	all +	2 nd <u>S</u>	1 st <u>A</u>
sin a - cos a - tan a +	sin a - cos a + tan a -	3 rd <u>T</u>	4 th <u>C</u>

This shows where the ratios are positive.

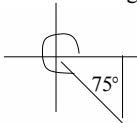
All Students Take Calculus or the CAST rule.

The reciprocal ratios are positive in the same quadrants as their primary counterparts.

Examples:

1. Find 2 angles (one positive and one negative) that are coterminal with 75° .

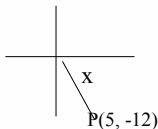
We are looking for angles with the same terminal arm.



$$435^\circ \text{ or } -285^\circ$$

add or subtract 360° .

2. The point (5, -12) creates an angle x in standard position. Find sin x, cos x and tan x.



$$x^2 + y^2 = r^2$$

$$(5)^2 + (-12)^2 = r^2$$

$$25 + 144 = r^2$$

$$r = 13$$

Find the radius of the circle that the point is on.

Solve for r (the radius of the circle).

The radius of the circle is positive.

$$\sin x = \frac{y}{r} \qquad \cos x = \frac{x}{r} \qquad \tan x = \frac{y}{x}$$

$$\sin x = \frac{-12}{13} \qquad \cos x = \frac{5}{13} \qquad \tan x = \frac{-12}{5}$$

Note: the signs of the ratios match the CAST rule.

3. If $\sin x = \frac{4}{5}$ find 2 values for $\cos x$.

$$\sin x = \frac{y}{r} \quad \therefore y = 4 \text{ and } r = 5$$

$$x^2 + y^2 = r^2$$

$$x^2 + 4^2 = 5^2$$

$$x = \pm 3$$

$$\therefore \cos x = \pm \frac{3}{5}$$

Use the circle equation to find x .

Substitute in the known values.

Solve for x

Sin x is positive so the quadrants are 1st

or

2nd.

In the 1st quadrant $\cos x$ is positive and in the 2nd quadrant $\cos x$ is negative.



iii) Radian Measure

While degrees measure the distance between the two arms of an angle, radian measure measures the distance around the circumference of the unit circle (or arc length) contained by the two arms of the angle.

If an angle in standard position has an arc length equal to the radius then the angle has a measure of 1 radian. In general $\theta = \frac{a}{r}$ where a is the arc length and r is the radius and θ is the angle in radians. Since one rotation is 360° and has an arc length of the entire circumference then $\theta = \frac{2\pi r}{r} = 2\pi$. So one rotation of $360^\circ = 2\pi$ radians.

$$\begin{aligned}\therefore 180^\circ &= \pi^r \\ \therefore 1^\circ &= \frac{\pi}{180^\circ}^r \text{ or } 1^r = \frac{180^\circ}{\pi}\end{aligned}$$

Note that 1^r is the equivalent of approximately 57.3°

Using these 2 equations we can convert from degrees to radians and back.

Examples:

1. Change to radians a) 150° b) -240°

a) $150\left(\frac{\pi}{180}\right) = \frac{5\pi}{6}$

Multiply by the equivalent of 1 degree and reduce.

b) $-240\left(\frac{\pi}{180}\right) = -\frac{4\pi}{3}$

Note that it is assumed to be in radians if π is in the answer.

2. Change to degrees

a) $\frac{3\pi}{4}$

b) $\frac{-\pi}{6}$

c) 2^r

a) $\frac{3\pi}{4}\left(\frac{180}{\pi}\right) = 135^\circ$

Multiply each of the fractions by the equivalent of 1 radian and reduce. Note it is assumed to be in radians if π is in the question. The superscript r indicates radians as well.

b) $\frac{-\pi}{6}\left(\frac{180}{\pi}\right) = -30^\circ$

c) $2^r\left(\frac{180}{\pi}\right) = \frac{360}{\pi} = 114.6^\circ$

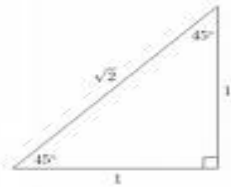
iv) Finding Exact Values Of Trig Ratios

Special Angles And The Unit Circle

Most trig ratios found on a calculator are approximate values because they are usually rounded off to four decimal places. Sometimes it is very important that the value for a trig ratio is *exact*. Before scientific calculators were readily available, mathematicians used special triangles showing the angles $\frac{\pi}{4}$ (45°), $\frac{\pi}{6}$ (30°), $\frac{\pi}{3}$ (60°) and the Unit Circle to calculate the trig ratios for several angles. The triangles and/or the unit circle show the exact values for the sine, cosine and tangent for the three angles above. The unit circle also gives these ratios for multiples of the three angles.

SPECIAL TRIANGLES

There are two special triangles. The first gives the values for the sides of an isosceles triangle with angles of 45° , 45° , and 90° .



This triangle is easy to remember as the short sides are each 1 unit in length. The hypotenuse can be calculated using Pythagorean Theorem.

$$\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$\tan 45^\circ = 1$ (note that this line has a slope of 1.
Tangent ratios are equal to the slope of the line with that angle.)

The second special triangle is created by bisecting an equilateral triangle (all sides equal to 2 and all angles 60°). This leaves a right angled triangle with angles of 30° and 60° .

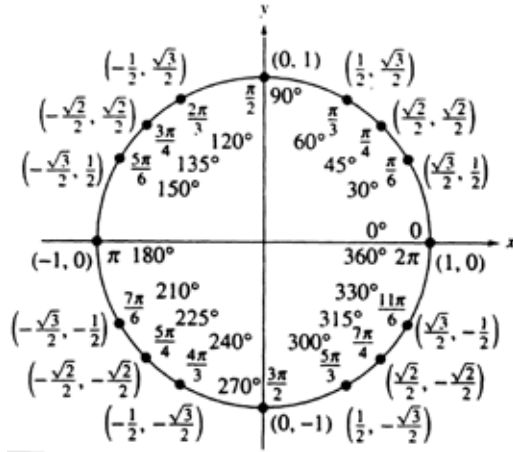


$$\sin 30^\circ = 1/2 \qquad \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2} \qquad \cos 60^\circ = 1/2$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} \qquad \tan 60^\circ = \frac{\sqrt{3}}{1} = \sqrt{3}$$

THE UNIT CIRCLE



The above circle shows the exact values for all of the 0° , 30° , 45° , 60° , and 90° angles and their equivalents in the other three quadrants. Each ordered pair (x, y) represents $(\cos x, \sin x)$ for that angle ie. $\cos 0^\circ = 1$, $\sin 0^\circ = 0$.

Note: The values for $\theta = \frac{\pi}{4}$ (45°) have been rationalized and now equal $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

To find tangent using the unit circle, divide the sin of the angle by the cosine of the angle

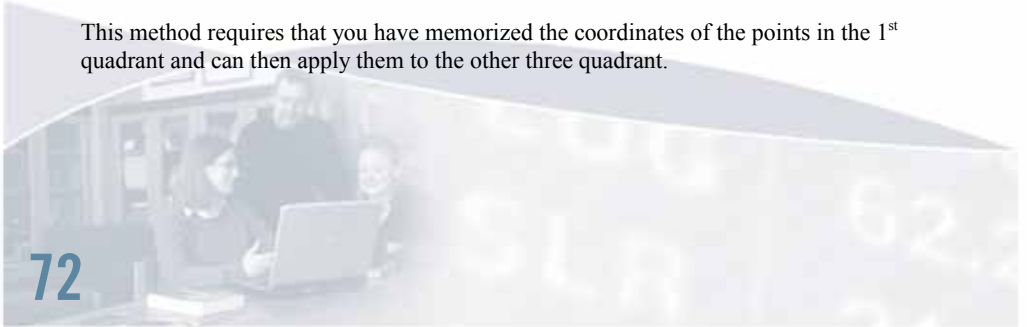
$$\text{ie. } \tan 45^\circ = \frac{\sin 45^\circ}{\cos 45^\circ} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2} \times \frac{2}{\sqrt{2}} = 1$$

Example:

1. Find $\cos \frac{5\pi}{6}$.

Method 1: Look on the unit circle. Find the angle indicated by $\frac{5\pi}{6}$. Remember that every point on the unit circle is $(\cos x, \sin x)$. Since the question asks for $\cos \frac{5\pi}{6}$ look at the first coordinate of the point and it is the $\cos \frac{5\pi}{6}$. $\therefore \cos \frac{5\pi}{6} = \frac{-\sqrt{3}}{2}$

This method requires that you have memorized the coordinates of the points in the 1st quadrant and can then apply them to the other three quadrant.



Method 2. Determine the angle's related acute angle.

Hint: Any radian measure with a denominator of 6 is related to the acute angle of 30° ($\frac{\pi}{6}$)

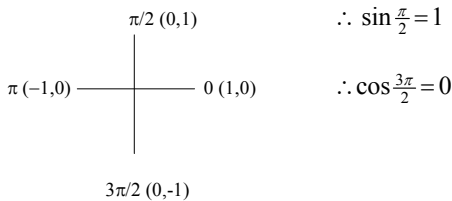
(similarly, radian measures with denominator 4 represent an acute angle of 45° ($\frac{\pi}{4}$) and a denominator of 3 represents an acute angle of 60° ($\frac{\pi}{3}$)).

The angle in the question is in the 2nd quadrant.

$$\begin{aligned}\cos \frac{5\pi}{6} &= -\cos \frac{\pi}{6} && \text{Use the triangles or the unit circle} \\ &= \frac{-\sqrt{3}}{2} && \text{Using the CAST rule.}\end{aligned}$$

(Cosine is negative in the 2nd quadrant and $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$)

For the angles on the x and y axes (quadrantal angles) the points (1,0), (0,1), (-1,0), (0,-1) are used from the unit circle. Remember every point on the unit circle is P (cos x, sin x)



Examples: Find the exact value for each of the following

1. $\sin \frac{5\pi}{3} = -\sin \frac{\pi}{3} = \frac{-\sqrt{3}}{2}$

RAA = $2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$

$\frac{5\pi}{3}$ is in the 4th quadrant and sin is negative using the CAST rule. Or look on the Unit Circle to find the angle and then its sine.

2. $\tan \frac{3\pi}{2}$

$$= \frac{\sin \frac{3\pi}{2}}{\cos \frac{3\pi}{2}} = \frac{-1}{0}$$

= Undefined

Quadrantal angle. Use the point P (0,-1)

3. $\cos(-\frac{\pi}{6}) = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

RAA = $\frac{\pi}{6}$, 4th quadrant angle and cosine is positive

4. $\csc \frac{5\pi}{4} = -\csc \frac{\pi}{4}$

$$= -\frac{\sqrt{2}}{1}$$

RAA = $\frac{5\pi}{4} - \pi = \frac{\pi}{4}$, (angle is in the 3rd quadrant)

Sine is negative and so is cosecant. Remember that cosecant is the reciprocal of sine. Look on the Unit Circle at $\frac{5\pi}{4}$ and then take the reciprocal of the y coordinate.

v) Using The Calculator To Find The Ratios

If the co-ordinates for the angle can not easily be determined by using the unit circle or the triangles then you must use your calculator to find the approximate value of the ratios. If no units are given it is assumed to be in radian measure. Scientific calculators have both radian and degree measure so make sure you read your manual to find out how to change from degree mode to radian mode. Check the instructions for your calculator on how to determine the trigonometric functions. When finding the values for the reciprocal functions use the $1/x$ key or the x^{-1} key on your calculator.

Examples:

Find

- $\sin 2^\circ = .9092$ Make sure your calculator is in radians.
- $\cos 55^\circ = .5736$ Change the calculator to degrees.
- $\csc 105^\circ = 1.0353$ Find $\sin 105^\circ$ then use the $1/x$ or x^{-1} key for \csc .
- $\cot 1.5^\circ = .071$ Find $\tan 1.5$ and use the $1/x$ key to find cotangent.

FINDING ANGLES - GIVEN THE VALUE OF THE RATIO

If you are given an exact value of a ratio from the unit circle then you are looking for the angle or angles that have that ratio. No approximations are allowed. Often a restriction on the angle(s) is given. If the restriction is in radians then the answer must be in radians. If the restriction is in degrees then the answer must be in degrees. Most often the question asks for the exact value of the angle and this is a clue that the unit circle or special triangles (see “Finding Exact Values of Trig Ratios”) must be used.

Examples:

- Find the exact value(s) for x if $0 \leq x \leq 2\pi$

a) $\sin x = \frac{\sqrt{3}}{2}$ Check the unit circle and find all angles that have the given sine value. The value is positive and so the angles are in the 1st and 2nd quadrants. The RAA that has the given value is $\frac{\pi}{3}$. The angles that have this RAA and are in the 1st and 2nd quadrants are $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. $x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$.

b) $\sec x = -\sqrt{2}$ Find the angles on the unit circle with a cosine of $-\frac{1}{\sqrt{2}}$, or use special triangles. Since $\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ then the RAA is $\frac{\pi}{4}$. Cosine and secant are negative in the 2nd and 3rd quadrants so the angles must be $\frac{3\pi}{4}$ or $\frac{5\pi}{4}$.

$$x = \frac{3\pi}{4} \text{ or } \frac{5\pi}{4}$$

If the exact value cannot be found then use the calculator to find an approximation of the angle either in radians or in degrees. Use the \sin^{-1} or \cos^{-1} or \tan^{-1} functions on the calculator to determine the value of the angle. Note that these functions are the inverses of the sine, cosine and tangent functions and not the reciprocal functions.

Examples:

1. Solve for x $0 \leq x \leq 360^\circ$

a) $\tan x = 3.2$ Make sure the calculator is in degrees and then use the $\tan^{-1}(3.2) = 72.6^\circ$ \tan^{-1} key. The RAA is 72.6° and since 3.2 is positive the angles we want are in the 1st and 3rd quadrants. The 1st quadrant angle is the RAA and to find the 3rd quadrant add 180° to the RAA.

$$x \approx 72.6^\circ \text{ or } 252.6^\circ$$

b) $\csc x = -1.2$ Change to $\sin x$ by using the $1/x$ or x^{-1} key.

$\sin x = -.8333$ Find the RAA by using $\sin^{-1}(.8\bar{3})$. Since the value is

$\sin^{-1}(.8\bar{3}) = 56.4^\circ$ negative the angles are in the 3rd and 4th quadrants. For the

3rd quadrant add 180° to the RAA and for the 4th quadrant angle subtract the RAA from 360° .

$$x = 236.4^\circ \text{ or } 303.6^\circ$$

2. Solve for x $0 \leq x \leq 2\pi$

$\cos x = -.3$ Change the calculator to radian measure. Find the RAA

$\cos^{-1}(.3) = 1.27$ using $\cos^{-1}(.3)$. The RAA is 1.27. Cosine is negative in

$x \approx \pi - 1.27$ or $\pi + 1.27$ the 2nd and 3rd quadrants so subtract the RAA from π and add it to π . The answers are assumed to be in radians if there are no units.

$$x \approx 1.87 \text{ or } 4.41$$

vi) Trigonometric Equations

An equation that involves one or more trigonometric ratios of a variable is called a trig equation. When solving a trig equation you are solving for the angle or angles that have the given ratios. Most of the equations are restricted to a range of x values (ie. $0 \leq x \leq 2\pi$) as the values of the trig ratios are continuous. The factoring that is used to solve some of these equations may be either common or trinomial.

Examples:

Find exact values for x , $0 \leq x \leq 2\pi$

- $\sin x = -\frac{1}{2}$
 $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

Since exact values are needed then you must use the unit circle and the 3rd and 4th quadrants as the sine is negative. $\sin \frac{\pi}{6} = \frac{1}{2}$ so the RAA is $\frac{\pi}{6}$. To find the angles add π to the RAA and subtract the RAA from 2π .
- $2 \sin^2 x - \sin x - 1 = 0$
 $2m^2 - m - 1 = 0$
 $(2m + 1)(m - 1) = 0$
 $(2 \sin x + 1)(\sin x - 1) = 0$
 $\therefore \sin x = -\frac{1}{2}$ or $\sin x = 1$
 $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$ or $x = \frac{\pi}{2}$

Looks like a trinomial equation. Let $m = \sin x$
 Factor as a difficult trinomial.
 Replace the m with $\sin x$.
 For the first equation check example 1 above.
 For the second equation use the unit circle and the $\sin \frac{\pi}{2} = 1$.
- $6 \cos^2 x + \cos x - 2 = 0$
 $(3 \cos x + 2)(2 \cos x - 1) = 0$

Factor as a difficult trinomial as question 2 above.
 You can let $\cos x = m$ if that helps to factor.

$\therefore 3 \cos x + 2 = 0$ or $2 \cos x - 1 = 0$
 $\cos x = -\frac{2}{3}$ or $\cos x = \frac{1}{2}$
 $\cos^{-1}(\frac{2}{3}) = .84$ or $\cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$
 $x \approx \pi - .84$ or $\pi + .84$ or $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$
 $x \approx 2.3$ or 3.98

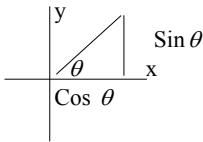
Use the calculator in radians to find the RAA. The angles are in the 2nd and 3rd quadrants. Add the RAA to π and subtract it from π to find the angles in the proper quadrants.

Use the unit circle to find the RAA. The angles are in the 1st and 4th quadrants. Use the RAA as the 1st quadrant angle and subtract it from 2π to find the 4th quadrant angle.

\therefore The exact solutions are $\frac{\pi}{3}, \frac{5\pi}{3}$ and the approximate solutions are 2.3, 3.98.

vii) Trigonometric Identities

Some equations in trigonometry are true for all angles not just specific ones. These equations are called identities. There are two identities that have been mentioned previously and these are used to prove others. On the unit circle each point $P(x) = (\cos \theta, \sin \theta)$ which leads to a pythagorean identity.



$$\sin^2 \theta + \cos^2 \theta = 1 \quad (1)$$

This can be rearranged to form 2 other useful identities.

$$\cos^2 \theta = 1 - \sin^2 \theta \quad (2)$$

$$\sin^2 \theta = 1 - \cos^2 \theta \quad (3)$$

The last identity used is obtained from the definition of $\tan \theta$.

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad (4)$$

All of the above four identities are true for all values of the angle. Use your calculator and try different angles for x to show that these are indeed true. There are four other identities that are also used. They are derived from the definitions of the reciprocal ratios. They are as follows:

$$\csc \theta = \frac{1}{\sin \theta} \quad (5)$$

$$\sec \theta = \frac{1}{\cos \theta} \quad (6)$$

$$\cot \theta = \frac{1}{\tan \theta} \quad (7)$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} \quad (8)$$

These eight identities are used to prove other statements that are true for all values of the angle. Note that $\sin^2 x$ means $(\sin x)(\sin x)$ or $(\sin x)^2$ but not $\sin x^2$ which is the square of the angle not the ratio. Use the preceding identities to prove the following questions. Where possible change all ratios to $\sin x$ or $\cos x$ and work on both the right and left sides until they are equal. Terms can not change sides nor can you multiply both sides by some amount. Any letter can be used to represent the angles. The most common are x and θ . Each question is split at the equal sign and by using the above identities as substitutions the question becomes the same on the right and the left sides.

Examples:

Prove the following:

$$1. \frac{\sin x}{\tan x} = \cos x$$

Split into Right Side and Left Side

$$\text{LS.} = \sin x \div \frac{\sin x}{\cos x}$$

$$\text{RS.} = \cos x \quad \text{Replace } \tan x \text{ with } \frac{\sin x}{\cos x}$$

$$= \sin x \left(\frac{\cos x}{\sin x} \right)$$

Invert and multiply and divide out the $\sin x$ in the numerator and denominator.

$$= \cos x$$

$\therefore \text{RS.} = \text{LS.}$ The identity is true. Q.E.D. is often used as the short form for the last sentence. It is Latin for "that was demonstrated".

$$2. \sin x = \frac{\tan x}{\sec x}$$

$$\text{LS.} = \sin x \quad \text{RS.} = \frac{\sin x}{\cos x} \div \frac{1}{\cos x}$$

Use identity (4) for $\tan x$ and (6) for $\sec x$.

$$= \left(\frac{\sin x}{\cos x} \right) \left(\frac{\cos x}{1} \right)$$

Invert and multiply. Reduce to lowest terms

$$= \sin x$$

$\therefore \text{LS.} = \text{RS.}$ Q.E.D.

$$3. (\sin x)(\cos x)(\tan x) = 1 - \cos^2 x$$

$$\text{LS.} = (\sin x)(\cos x) \left(\frac{\sin x}{\cos x} \right) \quad \text{RS.} = \sin^2 x \quad \text{Use identity (4) for } \tan x \text{ and (3) for the}$$

right side.

$$= \sin^2 x$$

$\therefore \text{RS.} = \text{LS.}$ Q.E.D.



$$4. \frac{1}{1 - \sin x} - \frac{1}{1 + \sin x} = \frac{2 \tan x}{\cos x}$$

Find a common denominator for the LS.

$$\text{LS.} = \frac{(1 + \sin x) - (1 - \sin x)}{(1 - \sin x)(1 + \sin x)} \quad \text{RS.} = \frac{2 \sin x}{\cos x} \div \cos x \quad \text{Replace } \tan x \text{ with identity (4).}$$

$$= \frac{2 \sin x}{1 - \sin^2 x} = \left(\frac{2 \sin x}{\cos x} \right) \left(\frac{1}{\cos x} \right) \quad \text{Collect like terms on the LS}$$

Multiply the denominator

LS.

$$= \frac{2 \sin x}{\cos^2 x} = \frac{2 \sin x}{\cos^2 x} \quad \text{Replace denominator LS with (2)}$$

\therefore LS. = RS. Q.E.D.

Identities are often used in solving some trigonometric equations.

Example:

Solve for x : $0 \leq x \leq 2\pi$

$$2 \cos x = 1 - \sin^2 x$$

Replace the RS. with $\cos^2 x$

$$2 \cos x = \cos^2 x$$

$$\cos^2 x - 2 \cos x = 0$$

Rearrange and set = 0. Common factor and solve as a quadratic equation. Set each factor = 0.

$$\cos x(\cos x - 2) = 0$$

$$\therefore \cos x = 0 \text{ or } \cos x = 2$$

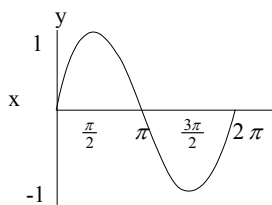
$$\therefore \cos^{-1}(0) \text{ or } \cos^{-1}(2)$$

$$= \frac{\pi}{2}, \frac{3\pi}{2} \quad = \text{no solution.}$$

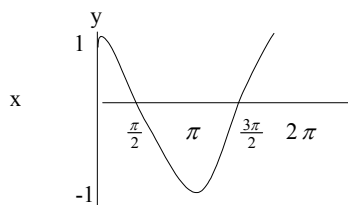
Values from the unit circle (No value for $\cos x$ is greater than one)

viii) Graphs of $y = \sin x$ and $y = \cos x$

The information on the unit circle for the values of $\sin x$ and $\cos x$ can be graphed on the x, y axes by plotting the values of the sine or cosine against their angle value. Both of these graphs are periodic as they repeat after every 2π . They repeat indefinitely in both directions. The graphs shown below are for one period of each of the graphs. Sine and cosine values lie between 1 and -1.



$y = \sin x$



$y = \cos x$



For Practice Problems on this section, please visit:
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