Pre-Calc Review

Co-ordinate systems, graphs & functions.

Question: Where did Cartesian coordinates come from?

- Claudius Ptolemy (~100 - 170 AD) - latitude & longitude
- Nicol Oresme (1325-1382) - lat + lon to draw graphs
- René Descartes (1597-1650) - cartesian

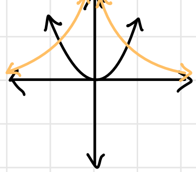
What is a function?

An assignment of an output to each possible input from some domain.

In calc, usually given by some rule or expression

ex. $f(x) = x^2$ $(x \in \mathbb{R})$ ↳ all points $(x, f(x)) = (x, x^2)$

$g(x) = \frac{1}{x^2}$ $(x \neq 0)$



Common functions

1° linear $f(x) = ax+b$

quadratic $g(x) = ax^2 + bx + c$

cubic $h(x) = ax^3 + bx^2 + cx + d$

In general, these are cases of polynomial functions

2° rational function: $r(x) = \frac{p(x)}{q(x)}$ p, q are polynomials

3° absolute value $f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$

4° Functions assembled out of powers like x^a where $a \in \mathbb{R}$ (usually restricted to $x > 0$)

ex. $x^{1/2} = \sqrt{x}$ makes no sense if $x < 0$ (unless you want complex numbers)

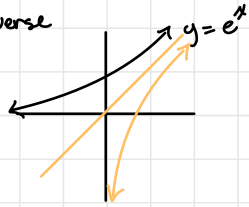
5° exponential functions $f(x) = a^x$ $(a>0)$ (wait for it!)
special case: e^x $e = 2.7182$

6° log functions (the inverses of exponential functions)
$f(x)$ has $g(x)$ as its inverse function if $f(g(x)) = g(f(x)) = x$

$\boxed{\text{ex.}}$ $f(x) = 1/x$ $(x \neq 0)$
$f(f(x)) = \frac{1}{1/x} = 1 \frac{x}{1} = x$

Notation: $f^{-1}(x)$ is usually used to denote inverse

DANGER: $f^{-1}(x) \neq \frac{1}{f(x)}$



$\log_a(x)$ is defined for $x>0$
$\log_a(a^x) = x$
Special case: $\underset{\text{natural log}}{\ln}(x) = \log_e(x)$

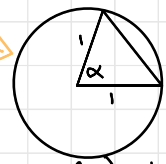
Aside: properties exponential fns & log fns

a^x:
0° $a^0 = 1$ (when $a>0$)
1° $a^x \cdot a^y = a^{x+y}$
2° $a^x / a^y = a^{x-y}$
3° $(a^x)^y = a^{xy}$
4° $a^{-x} = \frac{1}{a^x}$

log:
0° $\log_a(1) = 0$
1° $\log_a(a^{x+y}) = \log_a(a^x a^y)$
$\log_a(bc) = \log_a(b) + \log_a(c)$
2° $\log_a\left(\frac{b}{c}\right) = \log_a(b) - \log_a(c)$
3° $\log_a(b^c) = c\log_a(b)$
4° $\log_a\left(\frac{1}{b}\right) = \log_a(b^{-1})$
$= -\log_a(b)$

$\boxed{\text{Question:}}$ Where did the trig functions come from?

Claudius Ptolemy devised first one
chord(α) = length of the chord subtracted by α



Later, in India c 500-600 we get $\sin(x)$, $\cos(x)$, $\tan(x)$

$$\sin(x) = \frac{a}{c}$$

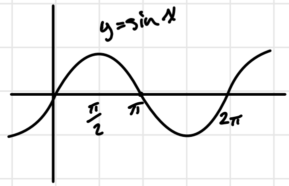
$$\cos(x) = \frac{b}{c}$$

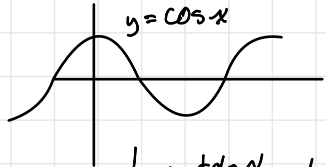
$$\tan(x) = \frac{\sin x}{\cos x} = \frac{a}{b}$$

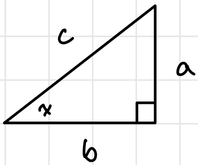
$$\cancel{\csc(x) = \frac{1}{\sin x}}$$

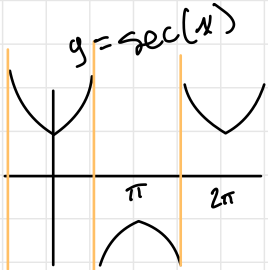
$$\sec(x) = \frac{1}{\cos x}$$

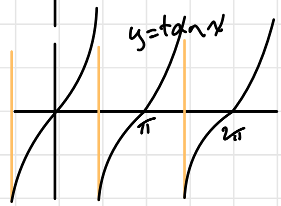
$$\cancel{\cot(x) = \frac{\cos x}{\sin x}}$$











Trig Formulas

$$\cos^2(x) + \sin^2(x) = 1$$

$$\sin(2x) = 2\sin x \cos x$$

$$\cos(2x) = 1 - 2\sin^2 x$$
$$= 2\cos^2 x - 1$$
$$= \cos^2 x - \sin^2 x$$

$$1 + \tan^2 x = \sec^2 x$$

Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\tanh(x) = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{sech}(x) = \frac{2}{e^x + e^{-x}} = \frac{1}{\cosh x}$$

$$\sinh^2 x + \cos^2 x = \left(\frac{e^x - e^{-x}}{2}\right)^2 + \left(\frac{e^x + e^{-x}}{2}\right)^2$$

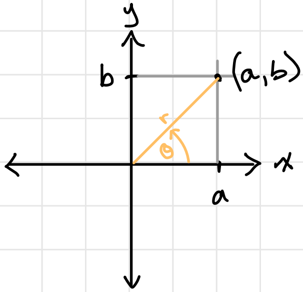
$$= \frac{(e^x)^2 + 2e^x(-e^{-x}) + (-e^{-x})^2}{2^2} + \frac{(e^x)^2 + 2e^x e^{-x} + (e^{-x})^2}{2^2}$$

$$= \frac{e^{2x} - \cancel{2e^0} + e^{-2x}}{4} + \frac{e^{2x} + \cancel{2e^0} + e^{-2x}}{4}$$

$$= \frac{1}{2} \cdot \frac{e^{2x} + e^{-2x}}{2} = \frac{1}{2} \cdot \cosh(2x)$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \dots$$

Sept 14: Polar Coordinates and inverse trig functions
(at least arctan) ← $\tan^{-1}$ on calculator



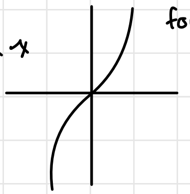
polar coordinates (r, θ)
vs. cartesian vectors $(x y)$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan \frac{y}{x} \qquad \cos\theta = \frac{x}{r}$$

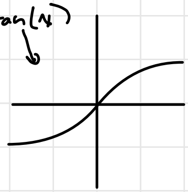
$$x = r\cos\theta$$

$$\tan\theta = y/x \qquad y = r\sin\theta$$

$y = \tan x$



invert for arctan(x)

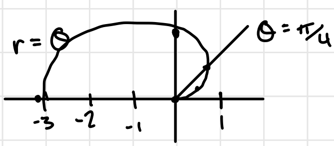


← only gives you value in first quadrant

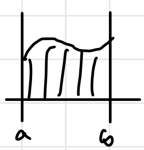
Functions in polar coordinates: $r = f(\theta)$

θ	r
0	0
$\pi/6$	$\pi/6$
$\pi/4$	$\pi/4$
$\pi/2$	$\pi/2$

ex. $r = \theta$

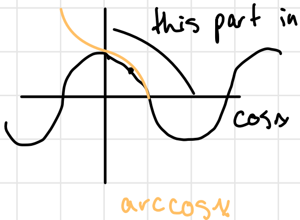


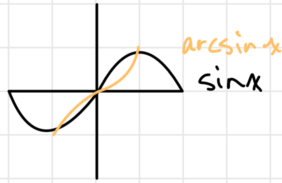
$$\int_a^b f(x)\,dx = \text{area}$$



$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt$$

We'll also encounter arcsin & arccos later...

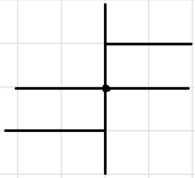




Limits

$\lim_{x \to a} f(x) = L$ means as x gets arbitrarily close to a, $f(x)$ gets close to L

$$f(x) = \frac{x}{|x|}$$



exception

$\lim_{x \to 0}$?

Limits (including some epsilonics)

"the limit of $f(x)$ as x approaches a is L"

$\lim\limits_{x \to a} f(x) = L$

This means: As x gets arbitrarily close to a, $f(x)$ gets arbitrarily close to L

We can get $f(x)$ as close as we would like to L, by geting x close enough to a

$\varepsilon > 0$) is how close we want $f(x)$ to be to L.

epsilon

ε

ie $|f(x) - L| < \varepsilon$ $\quad |2x+3-11|$

$= |2x - 8|$

$\delta > 0$ is how close we need to get x to a

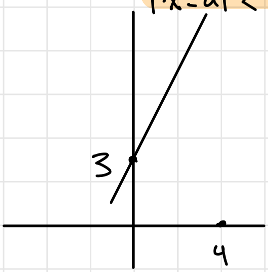
ie. $|x - a| < \delta$ $\quad |x-4| < \varepsilon$

Definition: $\lim\limits_{x \to a} = L$ Really means

for every $\varepsilon > 0$, there is a $\delta > 0$, such that if $|x - a| < \delta$, then $|f(x) - L| < \varepsilon$

$f(x) = 2x + 3$

$a = 4$



$L = 11$

$\lim\limits_{x \to 4} (2x+3) = 1$

Does this satisfy the ε-δ definition of limit?

Given an $\varepsilon > 0$

how do we find the necessary δ for that ε?

We want a $\delta > 0$, so that if $|x-4| < \delta$, then $|(2x+3)-11| < \varepsilon$

Reverse engineer the δ from

$|(2x+3)-11| < \varepsilon$

since each step is reversible

$|2x-8| < \varepsilon$

$2|x-4| < \varepsilon$

Whenever $|x-4| < \varepsilon/2$ we get $|(2x+3)-11| < \varepsilon$

So $\delta = \varepsilon/2$ does the job for $\varepsilon > 0$

$|x-4| = \dfrac{\varepsilon}{2}$

$\boxed{\text{ex.}}$ $f(x) = 2x$
$a = 51$

is $f(x)$ continuous at $a=51$

We have to check that for any $\varepsilon > 0$, there is a $\delta > 0$, such that if $|x-51| < \delta$ then
$|2x-102| < \varepsilon$

we do it for a generic $\varepsilon > 0$

Given $\varepsilon > 0$, how do we find the corresponding $\delta > 0$?

☆ note: δ will be defined in terms of ε

We'll use reverse-engineering

$\Leftrightarrow |2x-102| < \varepsilon$

$\Leftrightarrow 2|x-51| < \varepsilon$

$\Leftrightarrow |x-51| < \dfrac{\varepsilon}{2}$

So since each step is reversible,
if $|x-51| < \varepsilon/2$, then $|2x-102| < \delta$
$\therefore \delta = \varepsilon/2$ works

What about quadratics?

$f(x) = x^2$

$a = 0$

$L = F(0) = 0$

Given $\varepsilon > 0$, what is the corresponding $\delta > 0$?

$|x^2 - 0| < \varepsilon$

$\Rightarrow |x|^2 < 3$

$= |x| < \sqrt{3}$

$\Rightarrow |x - 0| < \sqrt{3}$

so $\delta = \sqrt{3}$ works

$a = 3$

$L = f(3) = 9$

$|x - 3| < \delta \Rightarrow |x^2 - 9| < \varepsilon$

$|x^2 - 9| < \varepsilon$

$\Rightarrow |(x+3)(x-3)| < \varepsilon$

$\Rightarrow |x - 3| < \dfrac{\varepsilon}{(x+3)}$

$\delta = \dfrac{\varepsilon}{(x+3)}$ ← divides by 0 at $x = -3$

two steps:

1. pick a $\# > 0$, that is less than the distance between 3 and -2 (2)
2. Then let $\delta = \min(2, \frac{\varepsilon}{8})$

how do we pick $\delta > 0$ so that we avoid $x = -3$ and have $\delta \leq \varepsilon$

So if $|x - 3| < \min(2, \frac{\varepsilon}{8})$

$= \dfrac{\varepsilon}{|x+3|} > \dfrac{\varepsilon}{8}$

so $|x - 3| < \varepsilon/8 < \dfrac{\varepsilon}{|x+3|}$

$= |x^2 - 9| < \varepsilon$

Practical Methods for Computing Limits Cont'd
(The squeeze theorem)

Rules for limits

1. $\lim_{x \to a} x^n = a^n$

2. $\lim_{x \to a} g(x) = g(a)$
   for most functions

3. $\lim_{x \to a} (f(x) + g(x)) = \left(\lim_{x \to a} f(x)\right) + \left(\lim_{x \to a} g(x)\right)$
   (provided the limits exist

4. $\lim_{x \to a} \left(f(x) \cdot g(x)\right) = \left(\lim_{x \to a} f(x)\right) \cdot \left(\lim_{x \to a} g(x)\right)$

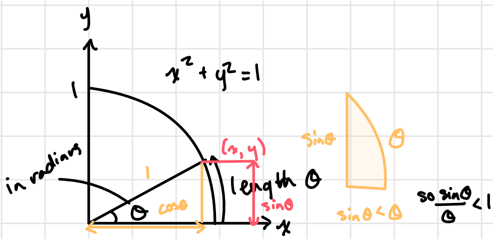
5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$
   provided all limits exist and $\lim_{x \to a} g(x) \neq 0$

6. $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$
   $= \lim_{t \to b} f(t)$ where $b = \lim_{x \to a} g(x)$
   ex. $\lim_{x \to 17} e^{\sin(x)} = e^{\sin(17)} = 0.382$

7. $\lim_{x \to a} c f(x) = c \lim_{x \to a} f(x)$
   if that limit exists

ex. 1 $\lim_{x \to 5} \dfrac{x^2 - 25}{x - 5} = \lim_{x \to 5} \dfrac{\cancel{(x-5)}(x+5)}{\cancel{x-5}} = 10$

ex. 2 $\lim_{x \to 0} \dfrac{\sin(x)}{x} = 1$ Why?
evaluate close to 0



Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$
for all x near a and $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} h(x)$ (which also exists)

then $\lim_{x \to a} g(x)$ exists & $= \lim_{x \to a} f(x) = \lim_{x \to 0} h(x)$

$f(x) \leq g(x) \leq h(x)$
as $x \to a$
L

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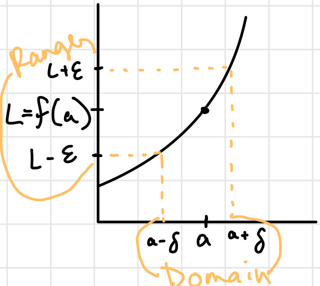
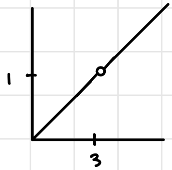


ε - epsilon - range (y)
δ - delta - domain (x)

$\lim\limits_{x\to a} f(x) = L$ for every $\varepsilon>0$ there is a $\delta>0$ such that $|x-a|<\delta$ implies $|f(x)-L|<\varepsilon$

* getting infinitely close to something w/o touching it

$$f(x) = \begin{cases} x/3, & x\neq 3 \\ 0, & x=3 \end{cases}$$



$\frac{|x-\overset{a}{3}|}{3} < \frac{\delta}{3}$ ← delta is really small number that makes this true

$|x/3 - 1| < \delta/3$

$|f(x)-L| < \delta/3$ $\qquad \varepsilon = \delta/3$

$|f(x)-L| < \varepsilon$

$\lim\limits_{x\to 3} f(x) = 3/3 = 1 = L$

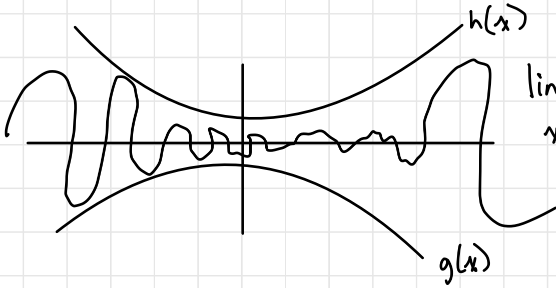
ex. $\lim\limits_{x\to(-1)} \frac{x^2+x}{x+1} = \frac{\cancel{(x+1)}x}{\cancel{x+1}} = x = -1$

$\lim\limits_{x\to\infty} \frac{\#}{x} = 0$

$\lim\limits_{x\to\infty} \frac{x}{\#} = \infty$

$\lim\limits_{x\to 1} \frac{x^2-1}{x-1} = \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}} = 2$

$\lim\limits_{x\to 2} \frac{x^2+x-12}{x-2} \frac{\times 1/x}{\times 1/x} = \lim\limits_{x\to 2} \frac{x+1-12/x}{1-2/x} =$



$$\lim_{x \to 0} f(x)$$

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} g(x) = 0$$

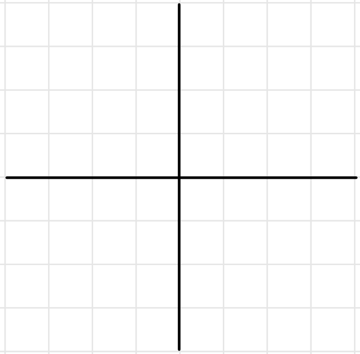
$$h(x) \geq f(x) \geq g(x)$$

$$\lim_{x \to 0} x^4 \cdot \cos\left(\frac{2}{x}\right)$$

$$x^4\, 1 \geq x^4 \cos\left(\frac{2}{x}\right) \geq -1\, x^4$$

$$x^4 \geq x^4 \cos\left(\frac{2}{x}\right) \geq -x^4$$

h(x) g(x)

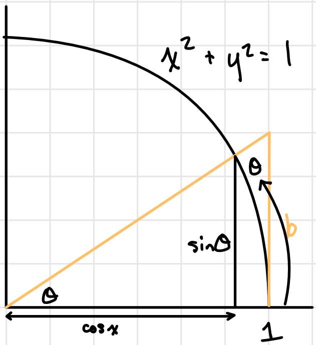


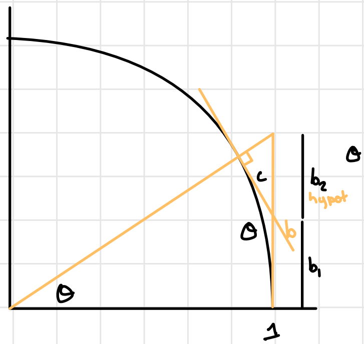
$$\lim_{x \to 0} x^4 = \lim_{x \to 0} -x^4$$

$$= 0 \qquad = 0$$

$$\therefore \lim_{x \to 0} x^4 \cdot \cos\left(2/x\right)$$

☆ for quiz #3, expect to see epsilonics.





$\theta < c + b_1 < b_2 + b_1 = b$

The Objective $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

(We'll need this limit later to sort out the derivatives of the basic trig functions)

$\frac{\sin\theta}{\theta}$ $\quad \sin\theta < \theta$ $\quad \sin\theta = \frac{opp}{hyp} = \frac{b}{\sqrt{1+b^2}}$

$$\frac{\sin\theta}{b} < \frac{\sin\theta}{\theta} < 1$$

as $\theta\to 0$, $b\to 0$: $\to 1$

$\frac{1}{\sqrt{1+b^2}}$

$$\lim_{\theta\to 0}\frac{1}{\sqrt{1+b^2}} = \lim_{b\to 0}\frac{1}{\sqrt{1+b^2}} = 1$$

$1 < \frac{\sin\theta}{\theta} < 1$

∴ so, by the squeeze theorem, $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

$$\lim_{x\to 0}\frac{\sin x + \cos x}{1/x}$$

$$= \lim_{x\to 0}(x\sin x + x\cos x)$$

$$= 0\sin 0 + 0\cos 0$$

$$= 0$$

$-2 \le \sin x + \cos x \le 2$

$\frac{1}{x} \to \infty$ as $x \to 0$

$$\frac{-2}{1/x} \le \frac{\sin x + \cos x}{1/x} \le \frac{2}{1/x}$$

$$-2x \le \sin x + \cos x \le 2x$$

$\to 0$

∴ $\lim_{x\to 0} = 0$

$$\lim_{x \to \sqrt{2}} \frac{x^2+3}{2x^2+4} = \frac{\sqrt{2}^2+3}{2(\sqrt{2})^2+4} = \frac{5}{8}$$

$$\lim_{x \to \infty} f(x) = L$$

means for all $\varepsilon > 0$, there is an $N > 0$, such that if $x > N$ then $|f(x) - L| < \varepsilon$

$< -N$

distance from origin

$$\lim_{x \to \infty} \frac{x^2+3}{2x^2+4} \times \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{1+\frac{3}{x^2}}{2+\frac{4}{x^2}}$$

go to 0 as $x \to \infty$

$$= \frac{1}{2}$$

denom smaller
$\therefore$ fraction bigger

$$\frac{x^2}{2x^2+4} \le \frac{x^2+3}{2x^2+4} \le \frac{x^2+3}{2x^2}$$

$$\frac{x^2}{2x^2+4} = \frac{1}{2+\frac{4}{x^2}}$$

$$\frac{x^2+3}{2x^2} = \frac{1}{2} + \frac{3}{2x^2}$$

$$\frac{1}{2} + \frac{x^2}{4}$$

as $x \to \infty$ $\quad 1/2$

Other Special Limits to memorize

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

$$\lim_{x \to \infty} \frac{e^{\sin(x)}}{x^2+2x+13} \to \infty$$

Since $-1 \le \sin x \le 1$

$$e^{-1} \le e^{\sin x} \le e^1$$

$$\frac{1}{e} \le e^{\sin x} \le e$$

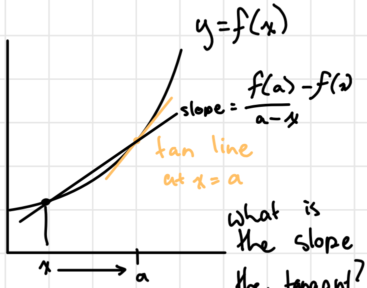
constant

$$\frac{\frac{1}{e}}{x^2+2x+13} \le \frac{e^{\sin x}}{x^2+2x+13} \le \frac{e}{x^2+2x+13}$$

constant

as $x \to \infty$

0



what is the slope of the tangent?

take the limit as $x \to a$ to get the slope of the tangent line

$$x = a+h$$

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \lim_{x \to a}$$

the derivative of f at $x=a$

$$\frac{f(x)-f(a)}{x-a}$$

$$= \lim_{h \to 0} \frac{f(a+h)-f(a)}{a+h-a}$$

$$= \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

$$\frac{dx}{dx} = 1$$

$$= \lim_{h \to 0} \frac{\cancel{(x+h)-x}}{\cancel{(x+h)-x}} = 1$$

$$f(x) = x^n \qquad (n > 2)$$

$$f'(x) = nx^{n-1}$$

$$\left.\frac{d}{dx} x^2\right|_7 = \lim_{h \to 0} \frac{(7+h)^2 - 7^2}{(\cancel{7}+h) - \cancel{7}}$$

$$= \lim_{h \to 0} \frac{\cancel{49} + 14h + h^2 - \cancel{7^2}}{h}$$

$$= \lim \frac{h(14+h)}{h}$$

$$= \lim 14+h$$

$$= 14$$

$$\frac{d}{dx} \sin x = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$\sin 2a = 2 \sin a \cos a$ is a special case of $\sin(a+b) = \sin a \cos b + \sin b \cos a$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x(\cos(h) - 1) + \sin(h)\cos(x)}{h}$$

$$= \lim_{h \to 0} \left[\sin x \underbrace{\frac{(\cos(h) - 1)}{h}}_{0} + \underbrace{\frac{\sin(h)}{h}}_{1} \cos x \right]$$

$$= \sin x(0) + 1 \cos x$$

$$= \cos x$$

Derivatives Cont'd

Recall: $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Rules

1) $\frac{d}{dx} x^n = nx^{n-1}$ - Power Rule

2) $\frac{d}{dx}(cf(x)) = c\left[\frac{d}{dx} f(x)\right]$

3) $\frac{d}{dx} f(g(x)) = (f \circ g)'(x) = f'(g(x)) \cdot g'(x)$ Chain Rule $\begin{array}{l} u = f(\omega) \\ \omega = g(x) \end{array}$ $\frac{du}{dx} = \frac{du}{d\omega} \cdot \frac{d\omega}{dx}$

4) $\frac{d}{dx}(f(x)g(x)) = f'(x) \cdot g(x) + f(x) \cdot g(x)$ Product Rule

5) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$ Quotient Rule

6.) $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$ Sum Rule

Derivatives of basic functions

1) $\frac{d}{dx} x^n = nx^{n-1}$

2) $\sin'(x) = \cos(x)$

$\cos'(x) = -\sin(x)$

3) $\frac{d}{dx}\tan(x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\left(\frac{d}{dx}\sin x\right) \cdot \cos x - \sin(x) \cdot \left(\frac{d}{dx}\cos x\right)}{(\cos x)^2} = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \left(\frac{1}{(\cos x)}\right)^2 = (\sec x)^2 = \sec^2 x$

4) $\frac{d}{dx}\sec(x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{d}{dx}(\cos x)^{-1} = \underset{\text{power}}{(-1) \cdot (\cos x)^{-2}} \cdot \underset{\text{chain}}{\frac{d}{dx}(\cos x)} = -(\cos x)^{-2} \cdot (-\sin x)$

$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec(x)\tan(x)$

5) $\frac{d}{dx} e^x = e^x$

$$= \frac{d}{dx}\left(e^{\ln(a)x}\right) = \frac{d}{dx} e^{\ln(a)\cdot x} = \left(\frac{d}{dw} e^{w}\right)\cdot \frac{dw}{dx} = e^{w}\cdot \frac{d}{dx}(\ln(a)\cdot x)$$

where $w = \ln a \cdot x$

$$= e^{\ln(a)\cdot x}\cdot \ln(a)\frac{d}{dx}x = a^{x}\cdot \ln(a)\cdot 1 = \ln(a)a^{x}$$

6) $\frac{d}{dx}a^{x} = \ln(a)\cdot a^{x}$

$(a>0)$

7) $\frac{d}{dx}\ln(x) = \frac{1}{x}$

$e^{\ln(x)} = x$

$\ln(e^{x}) = x$

$$\frac{d}{dx}e^{\ln(x)} = \frac{d}{dx}x = 1$$

$$\| $$

$$e^{\ln(x)}\cdot \frac{d}{dx}\ln(x) = x\cdot \frac{d}{dx}\ln(x)$$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

8) $\frac{d}{dx}\log_a(x) = \frac{1}{\ln(a)\cdot x}$

$(a>0)$

$$\frac{d}{dx}\left(\frac{\ln(x)}{\ln(a)}\right)$$

$$= \frac{1}{\ln(a)}\cdot\frac{d}{dx}\ln(x) = \frac{1}{\ln(a)}\cdot\frac{1}{x}$$

$$\frac{d}{dx}a^{\log_a(x)} = \frac{d}{dx}x = 1$$

$$\|$$

$$\ln\cdot \underbrace{a^{\log_a(x)}}_{x}\cdot\frac{d}{dx}\log_a(x)$$

Review & Chain Rule

Library of basic derivatives

0. $\frac{d}{dx}(x^n) = nx^{x-1}$
1. $\frac{d}{dx}\ln(x) = 1/x$
2. $\frac{d}{dx}e^x = e^x$
3. $\frac{d}{dx}\sin(x) = \cos(x)$
4. $\frac{d}{dx}\cos(x) = -\sin(x)$
5. $\frac{d}{dx}\tan(x) = \sec^2 x$
6. $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$

General Rules for Derivatives

→ Sum rule

→ Power rule

→ Product rule

→ Quotient rule

Chain Rule

Handle composition of functions

$\frac{d}{dx} f(g(x)) = f'(g(x)) \times g'(x)$

ex $\sqrt{\cos(x)}$

In Leibizian notation:

Let $u = f(x)$ & $v = g(x)$. Then the chain Rule is

$\frac{dy}{dy} = \frac{du}{dv} \cdot \frac{dv}{dx}$ $\qquad$ $\frac{d}{dx} = \frac{du}{dv} \cdot \frac{dv}{x}$

$f'(\overset{u}{g}(\overset{v}{x})) \cdot \overset{v}{g(x)}$

Examples

$\frac{d}{dx} e^{x^2} = \frac{d}{dx} e^x \Big|_{t=x^2} \cdot \frac{d}{dx} x^2$

$= e^x \ln x^2 \cdot 2x$

$= e^x \cdot 2x$

$= 2x e^x$

$\frac{d}{dx}\cos(\sin(x)) = \frac{d}{dx}\cos\Big|_{t=\sin x} \cdot \frac{d}{dx}\sin(x)$

$= -\sin(t) \cdot \cos x$

$= -\sin(\sin(x)) \cdot \cos(x)$

↳ as simple as it gets

$$\frac{d}{dx}\left(e^{\tan(\sqrt{x})}\right)^2 = \frac{d}{ds}s^2\Big|_{s=e^t}\cdot\frac{d}{dt}e^t\Big|_{t=\tan(u)}\cdot\frac{d}{dx}\tan(u)\Big|_{u=\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}$$

$$= 2s\Big|_{s=e^t}\cdot e^t\Big|_{t=\tan(u)}\cdot\sec^2(u)\Big|_{u=\sqrt{x}}$$

$2(e^{\tan(\sqrt{x})})\cdot e^{\tan(\sqrt{x})}\cdot\sec^2(\sqrt{x})\cdot$

$$= \cancel{2}(e^{\tan\sqrt{x}})\cdot e^{\tan\sqrt{x}}\cdot\sec^2\sqrt{x}\cdot\frac{1}{\cancel{2}\sqrt{x}}$$

$$= \frac{e^{2\tan\sqrt{x}}\cdot\sec^2(\sqrt{x})}{\sqrt{x}}$$

suppose $a>0$. What is $\frac{d}{dx}a^x$? (using chain rule)

We know $\frac{d}{dx}e^x = e^x$ and the chain rule?

$a = e^{\ln(a)}$ (since e^x & $\ln(x)$ undo eachother

so $a^x = (e^{\ln(a)})^x = e^{\ln(a)x}$

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{\ln(a)x} = \frac{d}{dt}e^t\Big|_{t=\ln(a)x}\cdot\frac{d}{dx}(\ln(a)x)$$

$$= e^x\Big|_{t=\ln(a)x}\cdot\ln(a)$$

$$= e^{\ln(a)x}\cdot\ln(a)$$

$$= a^x\cdot\ln(a)$$

Inverse Functions

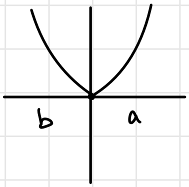
$f(x)$ & $g(x)$ are inverses (of each other)
if they undo each other
ie $f(g(x))=x$ & $g(f(x))=x$
(provided that the expressions make sense)
(might only be able to input a piece of the function.

ex. $f(x) = x^2$ The inverse function is

$g(x) = \sqrt{x}$ $x > 0$

$\downarrow$

$-\sqrt{x}$



★ often we can only invert part of a function at a time

ex. $f(x) = 3x + 5$ $g(x) = y$ $x = f(y)$

What is the inverse?

$x = f(y) = 3y + 5$ solve for y

$x - 5 = 3y$

$y = \frac{x-5}{3}$ $\therefore g(x) = \frac{x}{3} - \frac{5}{3}$

Notation: The inverse of a function $f(x)$ is usually denoted by $f^{-1}(x)$...

not to be confused w/ $(f(x))^{-1} = \frac{1}{f(x)}$

especially bad w trig func's

$\sin^2(x) = (\sin(x))^2$

but $\sin^{-1}(x)$ = inverse of $\sin(x)$

ambiguous $= \arcsin(x)$

$\frac{1}{\sin(x)} = \csc(x)$

A function $f(x)$ 1-1 (to something)

if $s \neq t \rightarrow f(s) \neq f(t)$

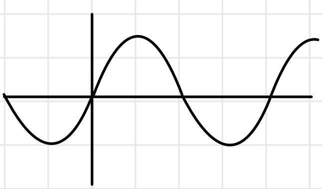
s & t in same interval

so for $f(x) = x^2$, it's 1-1 on $[-\infty, 0]$ & on $[0, \infty]$

but only in the interval (a,b) with $a < 0 < b$

What do we really mean by $\arcsin(x)$?

ie which piece of $\sin(x)$ are we inverting?



visits each value between $-1, 1$ infinitely on $(-\infty, \infty)$

The most we can hope to invert it on (because it's the largest an interval can be on which $\sin(x)$ is 1-1) is an interval of the form $(n\pi - \pi/2, n\pi + \pi/2)$

The one officially chosen for to be the part to invert is the part of $\sin(x)$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Oct 6 Seminar

$$f(x) = \sqrt{2x^2 - 5x + 2}$$
$$= (2x^2 - 5x + 2)^{1/2}$$
$$f'(x) = \frac{1}{2}(2x^2 - 5x + 2)^{-1/2} \cdot (4x - 5)$$
$$= \frac{4x - 5}{2(2x^2 - 5x + 2)^{1/2}} = \frac{4x-5}{2\sqrt{2x^2 - 5x + 2}}$$

$$g(x) = \sin(x)\cos(3x)$$
$$g'(x) = \cos(x)\cos(3x) + \sin(x)(-\sin(3x)) \cdot 3$$
$$= \cos(x)\cos(3x) - 3\sin(x)\sin(3x)$$

$$h(x) = \frac{e^x}{2x^3 + x}$$
$$= e^x(2x^3 + x)^{-1}$$

or quotient:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$$
$$\Rightarrow \frac{e^x(2x^3 + x) - (6x^2 + 1)e^x}{(2x^3 + x)^2}$$

$$h'(x) = e^x(2x^3 + x)^{-1} + e^x(-1)(2x^3 + x)^{-2} \cdot (6x^2 + 1)$$
$$= \frac{e^x}{2x^3 + x} + \frac{e^x(6x^2 + 1)}{(-1)\sqrt{2x^3 + x}}$$

Implicit Differentiation

$$\frac{d}{dx}(x^1) = 1 \cdot x^0 \cdot \frac{d}{dx}(x) = \frac{dx}{dx} = 1$$

$$\frac{d}{dx}(y) = 1 \cdot y^0 \cdot \frac{d}{dx}(y) = \frac{dy}{dx} = y'$$

$$\frac{d}{dy}(y) = 1$$

$$\frac{d}{dx}(x \cdot y) = \frac{d}{dx}(x) \cdot y + \frac{d}{dx}(y) \cdot x$$
$$= y + \frac{dy}{dx}x$$

$$\frac{d}{dy}(x \cdot y) = \frac{d}{dy}(x) \cdot y + \frac{d}{dy}(y) \cdot x$$
$$= \frac{dx}{dy} \cdot y + x$$
$$= x' \cdot y + x$$

$$\frac{d}{dx}(x^2 + y^2 = 1)$$
$$= \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$
$$= 2x + 2y(y') = 0$$

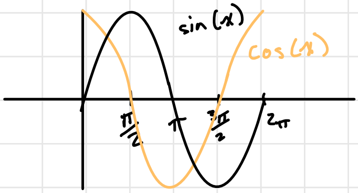
Oct 12: A brief look at implicit differentiation, then looking at the derivatives of inverse functions (maybe some optimization too?)

$$\sin(x+y) = 1 \Rightarrow \frac{d}{dx}\sin(x+y) = \frac{d}{dx}1 = 0$$

$$\frac{dy}{dx} = ?$$

$$= \cos(x+y)\frac{d}{dx}(x+y)$$
$$= \cos(x+y)\cdot\left(1+\frac{dy}{dx}\right)$$
$$= \cos(x+y) + \frac{dy}{dx}\cdot\cos(x+y)$$

$$\frac{dy}{dx} = \frac{-\cancel{\cos(x+y)}}{\cancel{\cos(x+y)}}$$
$$= -1$$



$$\sin(x+y) = 1$$
$$\Leftrightarrow x+y = \frac{\pi}{2} \pm n2\pi \quad (n \text{ is an integer})$$
$$\Rightarrow y = -x + \frac{\pi}{2} \pm 2\pi n$$
$$\Rightarrow \frac{dy}{dx} = -1$$

$$\frac{d}{dx}\ln(x) = \frac{d}{dx}\log_e(x)$$

$$e^{\ln x} = x \Rightarrow \frac{d}{dx}e^{\ln x} = \frac{d}{dx}x = 1 \qquad e^{\ln x}\cdot\frac{d}{dx}\ln x = 1 \Rightarrow \frac{d}{dx}\ln(x) = \frac{1}{e^{\ln x}} = \frac{1}{x} \checkmark$$

$(x>0)$

$$\boxed{\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

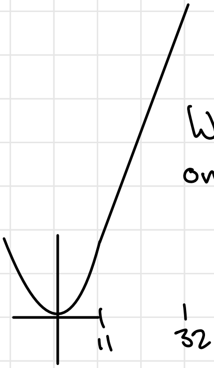
$$f(f^{-1}(x)) = x$$
$$\Rightarrow f'(f^{-1}(x))\cdot\frac{d}{dx}f^{-1}(x) = 1$$

$$\frac{d}{dx}\arcsin(x) = ?$$

$$\sin(\arcsin(x)) = x$$
$$\Rightarrow \frac{d}{dx}\sin(\arcsin(x)) = \frac{d}{dx}x = 1$$
$$\parallel$$
$$\cos(\arcsin(x))\cdot\frac{d}{dx}\arcsin(x)$$

$$\cos(t) = \sqrt{1-\sin^2(x)}$$

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\cos(\arcsin(x))}$$
$$= \frac{1}{\sqrt{1-\sin^2(\arcsin(x))}}$$
$$= \frac{1}{\sqrt{1-(\underbrace{\sin(\arcsin(x))}_{x})^2}}$$
$$= \frac{1}{\sqrt{1-x^2}}$$

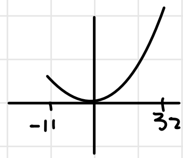


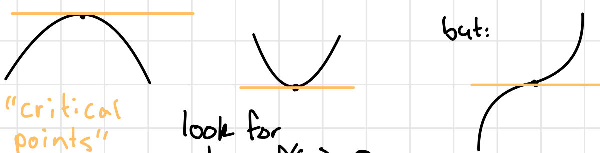
What is the minimum value of $f(x) = x^2$ on the interval $[11, 32]$

at $(x = 11)$

We know that $f(x)$ is increasing between 11 & 32, because $f'(x) = 2x > 0 \therefore$ increasing for all $x \geq 11$

What if the interval was $[-11, 32]$?





"critical points"

look for where $f'(x) = 0$

$f'(x) = 2x$

> 0 if $x > 0$

< 0 if $x < 0$

$\searrow \cup \nearrow \quad \therefore$ $x = 0$ is a minimum

$$f'(x) = \frac{d}{dx} x^2 = 2x = 0 \Leftrightarrow x = 0$$

Is $x = 0$ a max or a min or neither?

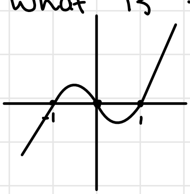
Alternatively, compare $f(0)$ with the values at the endpoints

$f(0) = 0^2 = 0$

$f(-11) = (-11)^2 = 121$

$f(32) = 32^2 = 1024$

What is the minimum value of $f(x) = x^3 - x$ on the interval $[-5, 5]$



$f'(x) = 3x^2 - 1 = 0 \Leftrightarrow x^2 = \frac{1}{3} \Leftrightarrow x = \pm\sqrt{\frac{1}{3}}$

$x = \frac{\pm 1}{\sqrt{3}}$

$f\left(\frac{-1}{\sqrt{3}}\right) = \left(\frac{-1}{\sqrt{3}}\right)^3 - \left(\frac{-1}{\sqrt{3}}\right) = \frac{-1}{3\sqrt{3}} - \frac{-1}{\sqrt{3}} = \frac{2}{3\sqrt{3}}$

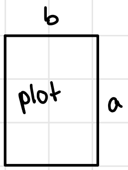
$f\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}}\right)^3 - \left(\frac{1}{\sqrt{3}}\right) = \frac{-2}{3\sqrt{3}}$

endpoints: $f(-5) = (-5)^3 - 5 = -125 + 5 = -120$ min

$f(5) = 5^3 - 5 = 120$ max

Oct 15/2023

A farmer has 100m of fencing and wants to enclose a rectangular plot with this fencing. What is the max area the farmer can enclose?



$A = ab$ ①

$P = 2a + 2b$ ②

$100 = 2a + 2b$

$50 - b = a$ ③

Sub ③ into ①:

$(50-b)\,b = A$

$-b^2 + 50b = A$

$0 \leq b \leq 50$

Find the critical points (in the interval)

Compare values of $A(w)$ of the critical points with the values at the endpoints.

$A' = -2b + 50$

Maximize A subject to $P = 100$

$A(0) = 0 \quad A(50) = 0$

$A'(w) = \frac{d}{dw}(50w - w^2)$

$= 50 - 2w$

$A'(w) = 0 = 50 - 2w$

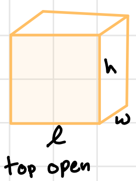
$\Leftrightarrow w = 25$

$A(25) = (50-25)\cdot 25 = 25^2 = 625$

$A_{max} = 625m^2$ $\bar{\omega}$ $w = l = 25$

Maximize the Volume of a rectangular box

$w = h$



$V = lwh$

$SA: \underset{\text{f+b}}{2lh} + \underset{\text{sides}}{2wh} + \underset{\text{bottom}}{lw} = 1m^2 = 10\,000\ cm^2$

$0 \leq w <$

need $l \geq 0$ so need $10\,000 - 2w^2 \geq 0$

ie $2w^2 \leq 10000$

$w^2 \leq 5000$

$w \leq \sqrt{5000}$

Isolate for l: $l(2h + w) + 2wh = 10000$

$\Rightarrow l = \frac{10000 - 2wh}{2h + w}$

$\Rightarrow l = \frac{10000 - 2w^2}{3w}$

$V = lwh = lw^2 = \frac{10000 - 2w^2}{3w} \cdot w^2$

$= \frac{10000w - 2w^3}{3}$

$$\frac{dV}{dw} = \frac{1}{3}\left(10000 - 6w^2\right) = 0$$

$$\Leftrightarrow 10000 - 6w^2 = 0$$

$$\Rightarrow w^2 = \frac{10000}{6}$$

$$\Rightarrow w = \sqrt{\frac{5000}{3}}$$

Sub in $w = 0$, $w = \sqrt{5000/3}$, $w = \sqrt{5000}$

into $V = \frac{1}{3}\left(10000w - 2w^3\right)$

& compare $V(0) = 0$

$$V(\sqrt{5000}) = \frac{1}{3}\left(10000(\sqrt{5000}) - 2(\sqrt{5000})^3\right)$$

$$= \frac{\sqrt{5000}}{3}\left(10000 - 2(\sqrt{5000})^2\right)$$

$$= 0$$

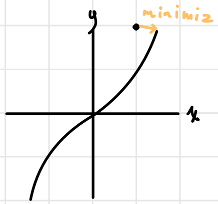
$$V\left(\sqrt{\frac{5000}{3}}\right) = \frac{1}{3}\left(10000 \cdot \sqrt{\frac{5000}{3}} - 2\left(\sqrt{\frac{5000}{3}}\right)^3\right)$$

$$= \frac{1}{3}\sqrt{\frac{5000}{3}}\left(10000 - 2\,\frac{5000}{3}\right)$$

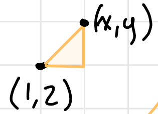
$$= \frac{1}{3}\sqrt{\frac{5000}{3}}\left(1000 - \frac{1000}{3}\right)$$

$$= \frac{2}{9}\sqrt{\frac{5000}{3}} \cdot 1000$$

What is the closest point on the curve $y = x^3$ to the point $(1, 2)$?



Distance between $(1,2)$ and (x,y) is $D = \sqrt{(x-1)^2 + (y-2)^2}$



$$= \sqrt{(x-1)^2 + (x^3-2)^2}$$

$$\frac{dD}{dx} = \frac{d}{dx}\left((x-1)^2 + (x^3-2)^2\right)^{1/2}$$

$$= \frac{1}{2}\left((x-1)^2 + (x^3-2)^2\right)^{-1/2} \cdot \left(2(x-1)\cdot 1 + 2(x^3-2)\cdot 3x^2\right)$$

$$= \frac{2x - 2 + 6x^5 - 12x^2}{2\sqrt{(x-1)^2 + (x^3-2)^2}}$$

$$= \frac{3x^5 - 6x + x - 1}{\sqrt{(x-1)^2 + (x^3-2)^2}}$$

To minimize D, we actually minimize $D^2 = (x-1)^2 + (y-2)^2$

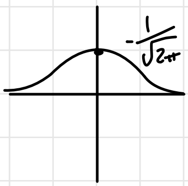
$$= (x-1)^2 + (x^3-2)^2$$

$$\frac{dD^2}{dx} = 2(x-1) + 2(x^3-2)\cdot 3x^2$$

$$= 2x - 2 + 6x^5 - 12x^2 = 2(x-1+3x^5-6x^2)$$

we still have the problem of finding where $\frac{dD^2}{dx} = 0$,

$y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ ← the standard "normal" density function



Find the absolute max & min of y on the real line

To check endpoints of $(-\infty, \infty)$, we take limits.

$$\lim_{x\to\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = 0$$

as $x \to \infty$, $-\frac{x^2}{2} \to -\infty$, so $e^{-x^2/2} \to 0$

$$\lim_{x\to-\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = 0$$

as $x \to -\infty$, $\frac{-x^2}{2} \to -\infty$, so $e^{-\frac{x^2}{2}} \to 0$

critical points:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot (-x)$$

$$= \frac{-x}{\sqrt{2\pi}} \underbrace{e^{-\frac{x^2}{2}}}_{>0} = 0 \text{ when } x = 0$$

$e^t > 0$ for all t

$$y(0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot 1$$

likely max

Check that $y(0) = \frac{1}{\sqrt{2\pi}}$ is a max:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{-x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right)$$

2nd derivative test

$$= \frac{1}{\sqrt{2\pi}}\left((-1)e^{-\frac{x^2}{2}} + (-x)(-x e^{-\frac{x^2}{2}})\right)$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \cdot (1-x^2)$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = \frac{-1}{\sqrt{2\pi}} < 0$$

$x=0$ is local max

1st derivative test

$$\frac{dy}{dx} = \frac{-x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

when $x < 0$, $\frac{dy}{dx} > 0 \to y$ increasing

when $x > 0$, $\frac{dy}{dx} < 0 \to y$ decreasing

∴ a maximum

↗ ↘

$\lim_{x \to 9} (3x+4) = 31$ $\longrightarrow$ means that B can force a win no matter what A does

ε - δ game:

A: Plays an $\varepsilon > 0$

B: Plays an $\delta > 0$ —— $\boxed{0 < \delta \leq \frac{\varepsilon}{3}}$

A: Plays an x w $|x-9| < \delta$

A wins if $|(3x+4)-31| \geq \varepsilon$

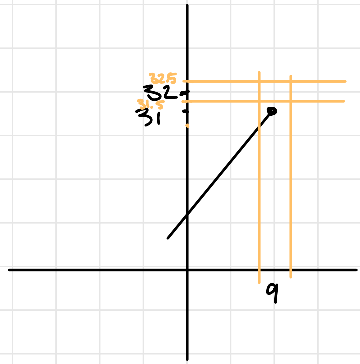
B wins if $|(3x+4)-31| < \varepsilon$

$|3x-27| < \varepsilon$

$|3(x-9)| < \varepsilon$

$|x-9| < \delta = \frac{\varepsilon}{3}$

$|x-9| < \delta$



$\lim_{x \to 9} (3x+4) = 36$

A wins if $|(3x+4)-32| \geq \varepsilon$

B wins if $|(3x+4)-32| < \varepsilon$

A plays $\varepsilon = \frac{1}{2}$

B $\delta > 0$

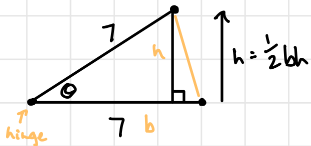
A: Pick x between $9-\delta$ and 9

then $3x+4 < 31 < 31.5$ so $|31.5-32| \geq 0.5$

Oct 19/2023

Maximize the area of the triangle.

$0 \le \theta \le \pi$



What is the area of the triangle?

$\sin\theta = \frac{opp}{hyp} = \frac{h}{7} \Rightarrow 7\sin\theta = h$

$\cos\theta = \frac{b}{7} \Rightarrow 7\cos\theta = b$

$A = \frac{1}{2}bh = \frac{49}{2}|\cos\theta|\sin\theta = \begin{cases} \frac{49}{2}\cos\theta\sin\theta & 0 \le \theta \le \frac{\pi}{2} \\ -\frac{49}{2}\cos\theta\sin\theta & \frac{\pi}{2} \le \theta \le \pi \end{cases}$

$A' = \frac{49}{2}\left((-\sin\theta)(\sin\theta) + (\cos\theta)(\cos\theta)\right)$

$= -\frac{49}{2}\sin^2\theta + \frac{49}{2}\cos^2\theta = 0?$

$\cos\theta = \sin\theta$ at $\theta = \frac{\pi}{4}$

$\cos\theta = -\sin\theta$ at $\theta = \frac{3\pi}{4}$

$\cos^2\theta = \sin^2\theta$

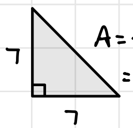
$A(0) = A(\pi) = 0$

$A(\frac{\pi}{4}) = \frac{49}{2}\cos\frac{\pi}{4}\sin\frac{\pi}{4}$

$= \frac{49}{4}$

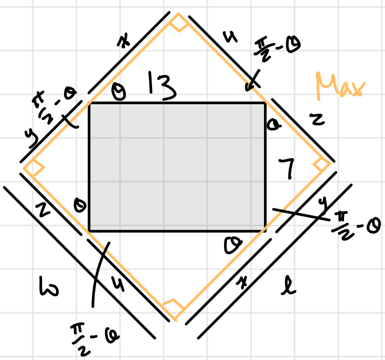
$A(\frac{3\pi}{4}) = \frac{49}{2}|\cos\frac{3\pi}{4}|\sin\frac{3\pi}{4}$

$= \left|\frac{49}{2}\left(\frac{-1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\right)\right| = \left|\frac{-49}{4}\right| = \frac{49}{4}$



$A = \frac{1}{2}bh$

$= \frac{49}{2}$



for $0 \le \theta \le \frac{\pi}{2}$

Max area of rectangle?

$A = lw = (x+y)(u+z)$

$= \left(13\cos\theta + \cancel{7\cos(\tfrac{\pi}{2}-\theta)}^{7\sin\theta}\right)\left(13\sin\theta + \cancel{7\sin(\tfrac{\pi}{2}-\theta)}^{7\cos\theta}\right)$

$= 13^2\cos\theta\sin\theta + 7^2\sin\theta\cos\theta$

$= 169\cos\theta\sin\theta + 49\sin\theta\cos\theta + 13\cdot7\cos^2\theta + 13\cdot7\cdot\sin^2\theta$

$= (13^2 + 7^2)\frac{1}{2}\sin(2\theta) + 13\cdot7(\underbrace{\cos^2\theta + \sin^2\theta}_{=1})$

$A = 109\sin(2\theta) + 91$

$A' = 109\cos(2\theta)\cdot 2$

$= 218\cos(2\theta)$

$\cos(2\theta) = 0$

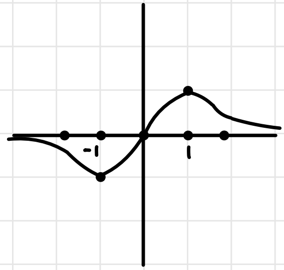
$2\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$

$A(\frac{\pi}{2}) = 109 + 91 = 200$

Qualitative Analysis (Curve Sketching: Chapter 5)

What can we say about $y = \dfrac{x}{1+x^2}$?
(without using software to draw picture)



__Checklist__

1. Intercepts: y - int $(x=0)$ $\quad y = \dfrac{0}{1+0^2} = 0$

   x - int $(y=0)$ $\quad 0 = \dfrac{x}{1+x^2} \Rightarrow x = 0$

2. Vertical Asymptotes: Look at points where def'n of y fails

   Since $y = \frac{x}{1+x^2}$ is defined and continuous for all x $\therefore$ y has no vertical asymptotes

3. Horizontal Asymptote $\lim\limits_{x\to\infty} y$ & $\lim\limits_{x\to -\infty} y$

$$\lim_{x\to\infty} \frac{x \to \infty}{1+x^2 \to \infty} = \lim_{x\to\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}(1+x^2)}$$

L'Hopital's Rule

If $f(x) \to 0$ as $x \to a$
& $g(x) \to 0$
or $f(x) \to \pm\infty$ as $x \to a$
$g(x) \to \pm\infty$
then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$

4. Critical points/increase/decrease/max/min

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{1+x^2}\right) = \frac{(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} = 0 \to x = \pm 1$$

y is increasing when $\frac{dy}{dx} > 0$
& decreasing when $\frac{dy}{dx} < 0$

$$\frac{dy}{dx} = \frac{1-x^2}{dx} \gtrless 0 \Leftrightarrow 1-x^2 > 0 \to x^2 < 1 \to -1 < x < 1$$

$$< 0 \to x^2 > 1 \to x > 1 \text{ or } x < -1$$

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\downarrow$	min	$\uparrow$	max	$\downarrow$

5. Curvature/Inflection points

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1-x^2}{(1+x^2)^2}\right)$$

$$= \frac{-2x(1+x^2)^2 - (1-x^2)2(1+x^2)(2x)}{(1+x^2)^4}$$

$$= \frac{-2x\left(1+x^2+2(1-x^2)\right)}{(1+x^2)^3}$$

$$= \frac{-2x-2x^3-4x+4x^3}{(1+x^2)^3}$$

$$= \frac{2x^3-6x}{(1+x^2)^3} = \frac{2x(x^2-3)}{(1+x^2)^3}$$

Potential Reflection points when $\frac{d^2y}{dx^2} = 0$

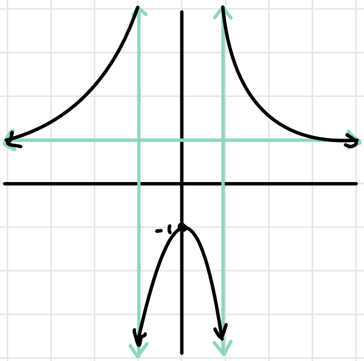
$$\frac{2x(x^2-3)}{(1+x^2)^3} = 0 \text{ when } x=0 \text{ or } x = \pm\sqrt{3}$$

>0 / <0 $\Leftrightarrow$ $2x(x^2-3) \geq 0$ / ≤ 0

x	$(-\infty,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},0)$	0	$(0,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},\infty)$
$f''(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\frown$	infl.	$\smile$	infl	$\frown$	infl	$\smile$

Curve Sketching

$f(x) = \frac{x^2+1}{x^2-1}$ Do the qualitative analysis



0. Domain (where $f(x)$ is defined)

all x s.t. $x^2-1 \neq 0$ ie. $x \neq 1, -1$

1. Intercepts

y-int ($x=0$) $f(0) = \frac{0^2+1}{0^2-1} = -1$

x-int ($y=0$) $0 = \frac{x^2+1}{x^2-1} \Rightarrow 0 = x^2+1 = 0$ $\therefore$ no x-int

2. Vertical Asymptotes

- look at where the denominator is 0, $x^2-1=0 \Rightarrow x = \pm 1$

check for VA's by taking limits from each side

$\lim_{x \to -1^-} \frac{x^2+1 \to 2}{x^2-1 \to 0^+} = +\infty$ $\quad$ $\lim_{x \to 1^-} \frac{x^2+1 \to 2}{x^2-1 \to 0^-} = -\infty$

$\lim_{x \to -1^+} \frac{x^2+1 \to 2}{x^2-1 \to 0^-} = -\infty$ $\quad$ $\lim_{x \to 1^+} \frac{x^2+1 \to 2}{x^2-1 \to 0^+} = +\infty$

3. Horizontal Asymptotes

$\lim_{x \to -\infty} \frac{x^2+1 \to \infty}{x^2-1 \to \infty}$

$= \lim_{x \to -\infty} \frac{2x}{2x}$

$= 1$

$\lim_{x \to \infty} \frac{x^2+1}{x^2-1} = \lim_{x \to \infty} \frac{x^2+1}{x^2-1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$= \lim_{x \to \infty} \frac{1 + \frac{1}{x^2} \to 0^+ \to 1^+}{1 - \frac{1}{x^2} \to 0^+ \to 1^-}$

$= 1^+$

4. critical pts/inc/dec/max/min

$$f'(x) = \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$= \frac{2x(x^2-1) - (x^2+1)2x}{(x^2-1)^2}$$

$$= \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2-1)^2}$$

$$= \frac{-4x}{(x^2-1)^2}$$

$=0 \Leftrightarrow -4x = 0 \Leftrightarrow x = 0$

$<0 \Leftrightarrow -4x < 0 \Leftrightarrow x > 0$

$>0 \Leftrightarrow -4x > 0 \Leftrightarrow x < 0$

5. Inflection pts & curvature

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-4x}{(x^2-1)^2}\right)$$

$$= \frac{-4(x^2-1)^{\cancel{2}\,1} - (-4x)(2)\cancel{(x^2-1)}(2x)}{(x-1)^{\cancel{4}\,3}}$$

$$= \frac{-4x^2+4+16x^2}{(x^2-1)^3} = \frac{4+12x^2}{(x^2-1)^3} = \frac{4(1+3x^2)}{(x^2-1)^3}$$

$$f''(x) = \frac{4(1+3x^2)}{(x^2-1)^3}$$

$\therefore$ undefined at $x = \pm 1$

$f''(x) > 0$ when $x^2 - 1 > 0$

↳ $x^2 > 1$

↳ $x > 1$ or $x < -1$

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
$f''(x)$	$+$	undef	$-$	undef	$+$
$f(x)$	$\cup$		$\cap$		$\cup$

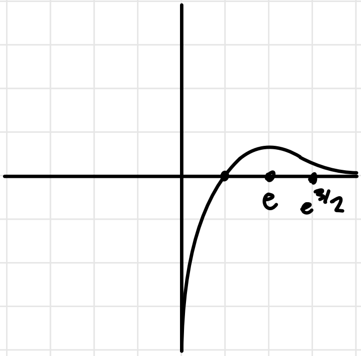
One more: $f(x) = \frac{\ln(x)}{x}$

0. Domain: $x > 0$

1. Intercepts no y-int, since $x \neq 0$ x-int $(y=0)$: $\frac{\ln(x)}{x} = 0 \to \ln x = 0$, $x = 1$

2. VA's $f(x)$ is defined &cts for $x > 0$, so the only place for a VA at $x = 0$

$$\lim_{x \to 0^+} \frac{\ln(x) \to -\infty}{x \to 0^+} = -\infty$$



3. HA l'hopitals rule

$$\lim_{x \to \infty^+} \frac{\ln(x) \to \infty}{x \to \infty} = \lim_{x \to \infty} \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx} x} = \lim_{x \to \infty} \frac{1/x \to 0^+}{1 \to 1} = 0^+$$

4. crit pts/inc/dec/max/min

$$f'(x) = \frac{d}{dx}\left(\frac{\ln(x)}{x}\right) = \frac{\frac{1}{x} \cdot x - \ln(x) \cdot 1}{x^2} = \frac{1 - \ln(x)}{x^2}$$

$= 0$ when $1 - \ln(x) = 0$
$\Leftrightarrow \ln(x) = 1$
$\Leftrightarrow x = e$

> 0 when $1 - \ln(x) > 0$
$\ln(x) < 1$
$x < e$

< 0 when $x > e$

x	$0, e$	e	e, ∞
$f'(x)$	+	0	−
$f(x)$	↑	max	↓

5. Inflection pts/curvature

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1 - \ln x}{x^2}\right) = \frac{-\frac{1}{x} \cdot x^2 - (1 - (\ln(x))) \cdot 2x}{x^4} = \frac{-x - 2x + 2x\ln(x)}{x^4}$$

$$= \frac{-3 + 2\ln(x)}{x^3}$$

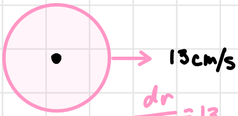
x	$0, e^{3/2}$	$e^{3/2}$	$e^{3/2}, \infty$
$f''(x)$	−	0	+
$f(x)$	⌒	undef	⌣

$= 0 \Leftrightarrow 2\ln(x) = 3$
$\Leftrightarrow \ln(x) = 3/2$
$\Leftrightarrow x = e^{3/2}$

$> 0 \Leftrightarrow 2\ln(x) > 3$
$x > e^{3/2}$
$< 0 \Leftrightarrow x < e^{3/2}$

Related Rates

Ex. 1 Drop a pebble into a pool, circular ripple which moves outward at 13 cm/s.



$c = 2\pi r$ $\quad \frac{dr}{dt} = 13$

a) How is the perimeter changing after 6.5s?

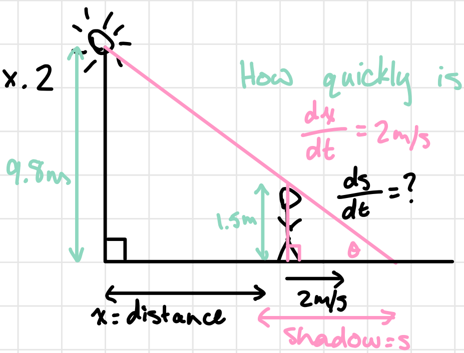
$$\frac{dc}{dt} = \frac{d}{dt}(2\pi r) = 2\pi \frac{dr}{dt} = 2\pi \cdot 13 = 26\pi \text{ cm/s}$$

b) how is the area enclosed by the perimeter changing after $\frac{13}{2} = 6.5$s?

$$A = \pi r^2 \qquad \frac{dA}{dt} = \frac{d}{dt}(\pi r^2) = 2\pi r \cdot \frac{dr}{dt} = 2\pi \cdot \underbrace{\left(13 \cdot 13/2\right)}_{r \text{ at } t = 6.5s} \cdot 13$$

$$= 13^3 \pi \text{ cm}^2/\text{s}$$

Ex. 2



How quickly is the shadow moving?

$$\tan\theta = \frac{1.5}{s} = \frac{9.8}{x+s}$$

$$9.8s = 1.5(x+s)$$

$$8.3s = 1.5x$$

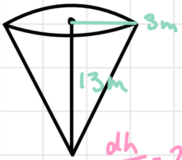
$$s = \frac{1.5}{8.3}x$$

$$\frac{ds}{dt} = \frac{1.5}{8.3} \cdot \frac{dx}{dt}$$

$$= \frac{1.5}{8.3} \cdot 2$$

$$= \frac{3}{8.3} \text{ m/s}$$

Ex.3 A conical tank $(r=3m,\ h=13m)$ is placed point down and water is poured in at $100\,mL/s = 0.1\,L/s$. How is the level of the water in the tank rising at the instant that the water in the tank has a max. depth of $7m$?



$\frac{dh}{dt} = ?$

know: $\frac{dV}{dt} = 100\,mL/s$

$V_{\text{cone w rad } r \text{ and height } h} = \frac{1}{3}\pi r^2 h$

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{3}\pi r^2 h\right) = \frac{1}{3}\pi\left(\frac{dr^2}{dh}\cdot h + r\cdot\frac{dh}{dt}\right)$$

$\frac{dr^2}{dh} = 2r\frac{dr}{dt}$

By similairity: $\frac{h}{r} = \frac{13}{3}$

$h = \frac{13r}{3}$ or $r = \frac{3h}{13}$

$h = 7 \Rightarrow r = \frac{21}{13}$

$$V = \frac{1}{3}\pi\left(\frac{3}{13}h\right)^2 h = \frac{3\pi}{169} = h^3$$

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt} = \frac{3\pi}{169}\cdot 3h^2\cdot\frac{dh}{dt}$$

$$100\,mL/s = \underbrace{\frac{3\pi}{169}\cdot 3\cdot 7^2}_{m^2}\cdot\frac{dh}{dt}$$

$1\,mL = \frac{1}{1000}L$

$= \frac{1}{1000}\cdot\frac{1}{1000}\,m^3$

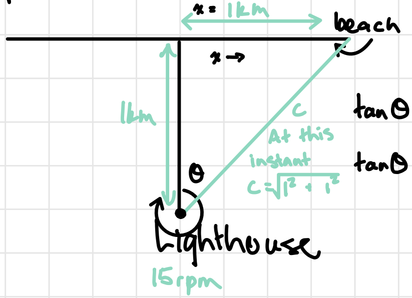
$= 10^{-6}\,m^3$

$$\frac{dh}{dt} = \frac{100\,mL/s}{\frac{3\pi}{169}\cdot 147\,m^2}$$

$$= \frac{10\times 10^{-6}\,\cancel{m^3}/s}{\frac{3\pi}{169}\cdot 147\,\cancel{m^2}}$$

One more example of Related rates, then integration!

Ex. A lighthouse is 1km from a straight beach. The lighthouse projects a beam of light that revolves at a constant rate of 15rpm. The beam encounters the beach when pointed that way. How is the point of contact moving along the beach when it is 1km from the closest point on the beach to the lighthouse?



$\frac{d\theta}{dt} = 15\text{rpm}$

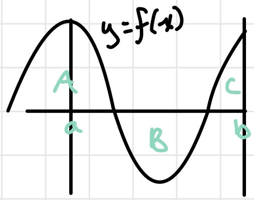
$\tan\theta = \frac{\text{opp}}{\text{adj}}$

$\tan\theta = x$

$$\frac{dx}{dt} = \frac{d}{dt}\tan\theta$$
$$= \left[\frac{d}{d\theta}\tan\theta\right]\frac{d\theta}{dt}$$
$$= \sec^2(\theta)\cdot\frac{d\theta}{dt}$$
$$= \frac{1}{\cos^2\theta}\cdot 15$$
$$= \frac{1}{(1/\sqrt{2})^2}\cdot 15$$
$$= 15\cdot 2 = 30\text{ km/h}$$

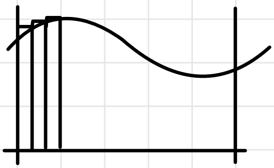
Differentiation is finding slopes of tangent lines

Integration is finding weighted areas between a graph and the x-axis



weighted area (for $a \le x \le b$)

$$= \int_a^b f(x)\,dx = A - B + C$$



How do we define this?

Properties of the definite integral

0° $\int_a^a f(x)\,dx = 0$

1° $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

2° $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx \rightarrow$ for any constant c

3° $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

4° $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

Fundamental Theorem of Calculus

I Suppose $F'(x) = f(x)$. Then $\int_a^b f(x)\,dx = F(b) - F(a)$

II Suppose we define $F(x) = \int_c^x f(t)\,dt$. Then $F'(x) = f(x)$ $\quad c \le t \le x$

Library of basic antiderivatives

1° $\int e^x\,dx = e^x + c$ because $\frac{d}{dx}(e^x+c) = e^x + 0 = e^x$

"indefinite integral" → generic integral

What about $\int a^x\,dx$? $(a>0)$

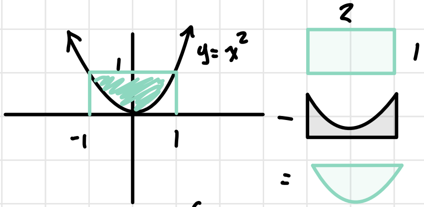
$= \frac{a^x}{\ln(a)} + c \rightarrow \frac{d}{dx}a^x = \ln(a)\cdot a^x \Rightarrow \frac{d}{dx}\left(\frac{a^x}{\ln(a)}\right) = a^x$ $\qquad a^x = e^{\ln(a)\cdot x}$

2° $\int \sin(x)\,dx = -\cos(x) + C \rightarrow \frac{d}{dx}(\sin(x)+c) = \cos(x)$

$\int \cos(x)\,dx = \sin(x) + C \rightarrow \frac{d}{dx}(-\cos(x)+c) = \sin(x)$

3° $\int \frac{1}{x}\,dx = \ln(x) + C \rightarrow \frac{d}{dx}\ln(x) = \frac{1}{x} = x^{-1}$

4° $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \rightarrow \frac{d}{dx}x^k = kx^{k-1}$



Area of rectangle $= 2 \cdot 1 = 2$

Area below curve $y = x^2$

$= \int_{-1}^{1} x^2\,dx = \left(\frac{x^3}{3} + C\right)\Big|_{-1}^{1}$

$= \left(\frac{1^3}{3} + \cancel{C}\right) - \left(\frac{(-1)^3}{3} + \cancel{C}\right)$

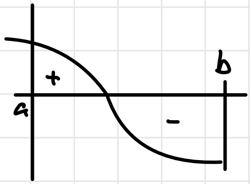
$= \frac{2}{3}$

Area $= 2 - \frac{2}{3} = \frac{6}{3} - \frac{2}{3} = \frac{4}{3}$

Quickie Recap

$\int_a^b f(x)\,dx$ definite integral

represents a weighted area



$\int f(x)\,dx = F(x) + C$ indefinite integral

means that $F'(x) = f(x)$

$\int_a^b f(x)\,dx = F(b) - F(a)$ by the fundamental theorem

Properties: 1. $\int f(x)\,dx$ & $\int_a^b f(x)\,dx$ respect addition and multiplication by constants

2. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

3. $\int_a^a f(x)\,dx = 0$

4. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

Rules & Basic Antiderivatives

1. Power Rule $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ if $n \neq -1$

$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln(x) + C$

2. $\int e^x\,dx = e^x + C$ $\quad \int a^x\,dx = \frac{a^x}{\ln(a)} + C$ $(a>0)$

3. $\int \cos(x)\,dx = \sin(x) + C$ $\quad \int \sec^2(x)\,dx = \tan(x) + C$

$\int \sin(x)\,dx = -\cos(x) + C$ $\quad \int \sec(x)\tan(x)\,dx = \sec(x) + C$

Recall The Chain Rule is

$$\frac{d}{dx}(f(g(x))) = (f\circ g)'(x) = f'(g(x))\cdot g'(x)$$

so $$\int f'(g(x))\cdot g'(x)\,dx = f(g(x)) + C$$

Substitution: Given $\int_a^b h(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} h(u)\,du.$

That is: we substitute u in for $g(x)$ and misuse $\frac{du}{dx} = g'(x)$ by rearranging it as $du = g'(x)\,dx$

$\boxed{\text{Ex.}}$ $\int 2\cos(2x)\,dx$ substitute $u = 2x$ so $du = 2dx$ (as $\frac{du}{dx} = \frac{d}{dx}(2x) = 2$)

$$= \int \cos(u)\,du = \sin(u) + C$$

$$= \sin(2x) + C$$

$$\int_0^{\pi} 2\cos(2x)\,dx = \sin(u) + C$$

$$= \sin(\pi) - \sin(0) = 0$$

$\boxed{\text{Ex.}}$ $\int_0^1 e^{3x}\,dx$

$$= \int_0^3 e^u \cdot \tfrac{1}{3}\,du = \tfrac{1}{3}e^u\Big|_0^3$$

$$= \tfrac{1}{3}e^3 - \tfrac{1}{3}e^0$$

$$= \tfrac{1}{3}(e^3 - 1)$$

$u = 3x$, so $\frac{du}{dx} = 3$, so $du = 3dx$, so $dx = \frac{1}{3}du$

x	u
0	0
1	3

or $$\int_{x=0}^{x=1} e^u \cdot \tfrac{1}{3}\,du = \tfrac{1}{3}e^u\Big|_{x=0}^{x=1} = \tfrac{1}{3}e^{3x}\Big|_0^1 = \tfrac{1}{3}(e^3 - 1)$$

$\boxed{\text{Ex.}}$ $\int x e^{x^2}\,dx = \int e^w \cdot 2\,dw$

$$= \tfrac{1}{2}e^w + C$$

$$= \tfrac{1}{2}e^{x^2} + C$$

$w = x^2$

$dw = \left(\frac{d}{dx}x^2\right)dx$

$= 2x\,dx$

$x\,dx = 2\,dw$

$\boxed{\text{Ex.}}$ $\displaystyle\int_0^4 \sqrt{x+1}\,dx = \int_1^5 \sqrt{u}\,du = \int_1^5 u^{1/2}\,du = \frac{u^{3/2}}{3/2}\Bigg|_1^5$

$u = x+1$

$du = dx$

x	u
0	1
4	5

$\displaystyle = \frac{2}{3}u^{3/2}\Big|_1^5 = \frac{2}{3}5^{3/2} - \frac{2}{3}1^{3/2}$

$\boxed{\text{Ex.}}$ $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx \rightarrow = \int \frac{1}{\sqrt{t}}\left(-\frac{1}{2}\right)dt$

$t = 1-x^2$

$\dfrac{dt}{dx} = -2x$

$dt = -2x\,(dx)$

$x\,dx = -\frac{1}{2}dt$

$\displaystyle = -\frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt$

$\displaystyle = -\frac{1}{2}\int t^{-1/2}\,dt$

$\displaystyle = \left(\frac{-1}{2}\right)\cdot\frac{t^{1/2}}{1/2} + C$

$= -t^{1/2} + C$

$= -\sqrt{1-x^2} + C$

Alternatively:

$\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$

$\displaystyle\int (-1)\,dw = -w + C$

$= -w + C$

$= -\sqrt{1-x^2} + C$

$w = \sqrt{1-x^2}$

$\dfrac{dw}{dx} = \dfrac{1}{\cancel{2}}(1-x^2)^{-1/2}\cdot(-\cancel{2}x)$

$(-)\,dw = \dfrac{x}{\sqrt{1-x^2}}$

$\boxed{\text{Ex.}}$ $\displaystyle\int \frac{e^{2x}}{\sqrt{1+e^x}}\,dx = \int \frac{u}{\sqrt{1+u}}\,du = \int \frac{w-1}{\sqrt{w}}\,dw = \int\left(\frac{w}{\sqrt{w}} - \frac{1}{\sqrt{w}}\right)dw$

$\displaystyle = \int\left(\sqrt{w} - \frac{1}{\sqrt{w}}\right)dw$

$\displaystyle = \int\left(w^{\frac{1}{2}} - w^{-\frac{1}{2}}\right)dw$

$\displaystyle = \frac{w^{3/2}}{3/2} - \frac{w^{1/2}}{1/2} + C$

$\displaystyle = \frac{2}{3}w^{3/2} - 2w^{1/2} + C$

$u = e^x$ $\qquad e^{2x} = (e^x)^2 = u^2$

$\frac{du}{dx} = e^x$ $\qquad w = 1+u$

$dw = du$

$du = e^x\,dx$ $\qquad u = w-1$

or: $w = e^x + 1$

$\displaystyle = \frac{2}{3}(u+1)^{3/2} - 2(1+u)^{1/2} + C$

$\displaystyle = \frac{2}{3}(1+e^x)^{3/2} - 2(1+e^x)^{1/2} + C$

or: $t = \sqrt{1+e^x}$ $\qquad \frac{dt}{dx} = \frac{1}{2}(1+e^x)^{-\frac{1}{2}}$ $\qquad t\,dt = \frac{dx}{2}$

$t^2 = 1+e^x$

$e^x = t^2 - 1$ $\qquad dx = 2\sqrt{1+e^x}\,dt$

$= 2t\,dt$

$\displaystyle\int \frac{e^{2x}}{\sqrt{1+e^x}}\,dx = \int (t^2-1)\,2\cdot dt = \frac{2}{3}t^3 - 2t + C$

$\displaystyle = \frac{2}{3}\left(\sqrt{1+e^x}\right)^3 - 2\sqrt{1+e^x} + C$

$\displaystyle = \frac{2}{3}(1+e^x)^{3/2} - 2(1+e^x)^{1/2} + C$

$\boxed{\text{Ex.}}$ $\displaystyle\int \sin(x)\cos(2x)\,dx = \int \sin x\,(2\cos^2 x - 1)\,dx$

$u = \cos(x)$ $\qquad \displaystyle = \int (2u^2-1)(-1)\,du$

$du = -\sin(x)\,dx$ $\qquad \displaystyle = \int (u - 2u^2)\,du$

so $\sin(x)\,dx = (-1)\,du$ $\qquad = u$

A bit more substitution and then "integration by parts"

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{\pi}{2}$$

using substitution

$x = \sin(t)$

x	t
-1	$-\frac{\pi}{2}$
1	$\frac{\pi}{2}$

$$\frac{dx}{dt} = \frac{d}{dt}\sin(t) = \cos(t)$$

$$dx = \cos(t)\,dt$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{1-\sin^2(t)} \cdot \cos(t)\,dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t)\cdot\cos(t)\,dt$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \cos^2(t)\,dt$$

$$\cos(2t) = 2\cos^2(t) - 1$$

$$\cos^2(t) = \frac{1+\cos(2t)}{2}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos(2t)}{2}\right) dt$$

$u = 2t$

$du = 2dt$

$dt = \frac{1}{2}du$

$$= \int_{t=-\pi/2}^{t=\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos(u)\right)\cdot\frac{1}{2}\,du$$

$$= \frac{1}{4}\int_{t=-\pi/2}^{t=\pi/2} (1+\cos(u))\,du$$

$$= \frac{1}{4}(u + \sin(u))\Big|_{t=-\pi/2}^{t=\pi/2}$$

$$= \frac{1}{4}(2t + \sin(2t))\Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{4}\left(\cancel{2}\frac{\pi}{\cancel{2}} + \sin\left(\cancel{2}\frac{\pi}{\cancel{2}}\right)\right) - \frac{1}{4}\left(\cancel{2}\left(\frac{-\pi}{\cancel{2}}\right) + \sin\left(\cancel{2}\left(\frac{-\pi}{\cancel{2}}\right)\right)\right)$$

$$= \frac{1}{4}\pi - \frac{1}{4}(-\pi)$$

$$= \frac{\pi}{2}$$

$u = f(x)$

$v = g(x)$

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + g'(x)f(x)$$

$$(u\cdot v)' = u'\cdot v + u\cdot v' \Rightarrow u\cdot v' = (u\cdot v)' - u'\cdot v$$

$$\Rightarrow \int u\cdot v'\,dx = \int (u\cdot v)'\,dx - \int u'\cdot v\,dx$$

$$= u\cdot v - \int u'\cdot v\,dx$$

$$\int_0^{13} x e^{-x}\, dx$$

$u =$ $\qquad v' = e^{x}$

$v' =$ $\qquad u = -e^{-x}$

↳ make this product simpler

$$= x(-e^{-x})\Big|_0^{13} - \int_0^{13} 1 \cdot (-e^{-x})\, dx$$

$$\underbrace{}_{u \cdot v} \qquad - \int u' \cdot v\, dx$$

$$= -xe^{-x}\Big|_0^{13} + \int_0^{13} e^{-x}\, dx$$

$$= (-13 \cdot e^{-13}) - \cancel{(-0e^{-0})} + (-e^{-0})\Big|_0^{13}$$

$$= -13e^{-13} + (-e^{-13}) - (-e^{-0})$$

$$= -13e^{-13} - e^{-13} + 1 = -14e^{-13} + 1$$

$$\int \arctan(x)\, dx = \int 1 \cdot \arctan(x)\, dx = x \arctan(x) - \int \frac{x}{1+x^2}\, dx$$

$u = \arctan(x)$ $\qquad v = 1$

$u' = \frac{1}{1+x^2}$ $\qquad v' = x$

$$\hookrightarrow = x \arctan(x) - \int \frac{1}{w} \cdot \frac{1}{2}\, dw$$

$$= x \arctan(x) - \frac{1}{2}\ln(w) + C$$

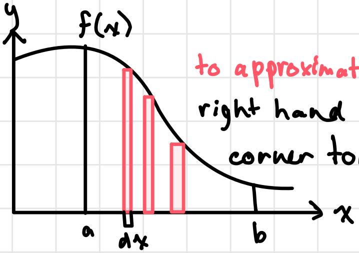
$$= x \arctan x - \frac{1}{2}\ln(1+x^2) + C$$

$w = 1 + x^2$

$dw = 2x\, dx$

$\frac{1}{2} dw = x\, dx$

Seminar Nov 24th



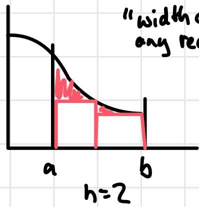
to approximate

right hand rule: right corner touching line

$$\int_a^b f(x) = \lim_{n\to\infty} \underbrace{\left(\frac{b-a}{n}\right)}_{dx}\underbrace{\sum f\left(a+\left(\frac{b-a}{n}\right)c\right)}_{dy,\ \text{adding all together}}$$

"width of any rectangle"

"height of rectangles"



$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$i^2 = \frac{n(n+1)(2n^2+1)}{8}$$

i, n = var (i, n')

sigma = sum (___ , i, 1, n)

lim $\left(\left(\frac{b-a}{n}\right)\right.$ sigma, n = infinity)

$$\int_1^4 3x^2\,dx = \lim_{n\to\infty}\left(\frac{3}{n}\right)\sum_{i=1}^{n} f\left(1+\left(\frac{3}{n}\right)i\right)$$

$$3\sum_{i=1}^{n}\left(1+\left(\frac{3}{n}\right)i\right)^2 = 3\sum_{i=1}^{n}\left(1+\left(\frac{6}{n}\right)i+\frac{9}{n^2}i^2\right)$$

$$= 3\left[\sum_{i=1}^{n}1+\sum_{i=1}^{n}\left(\frac{6}{n}\right)i+\sum_{i=1}^{n}\left(\frac{9}{n^2}\right)i^2\right]$$

$$= 3\left(n+\frac{6}{n}\sum_{i=1}^{n} i+\frac{9}{n^2}\sum_{i=1}^{n} i^2\right)$$

$$= 3\left(n+\frac{\overset{3}{\cancel{6}}}{\cancel{n}}\left(\cancel{n}\frac{n+1}{\cancel{2}}\right)+\frac{9}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right)$$

$$= 3\left(n+3(n+1)+\frac{3}{2n}(2n^2+3n+1)\right)$$

$$= 3\left(\frac{2n(n)}{2n}+\frac{2n(3)(n+1)}{2n}+\frac{3(2n^2+3n+1)}{2n}\right)$$

$$= 3\,\frac{6n^2+6n^2+6n+6n^2+9n+3}{2n}$$

$$= 3\,\frac{14n^2+15n+3}{2n}$$

$$= \lim_{n\to\infty}\left(\frac{3}{n}\right)\left(\frac{3}{2n}\right)\left(14n^2+15n+3\right)$$

$$= \lim_{n\to\infty}\frac{9(14\cancel{n^2})}{2\cancel{n^2}}+\cancel{\frac{9(15n)}{2n^2}}^{\,0}+\cancel{\frac{27}{2n^2}}^{\,0}$$

$$= \lim_{n\to\infty}\frac{9(14)}{2} = 63$$

$$\int \ln(x^2)\,dx = \int 2\ln(x)\,dx = 2\int \ln(x)\,dx = 2\left[x\ln(x) - \int \frac{1}{x} x\,dx\right]$$

$u = \ln x \qquad v' = 1$

$$= 2\left[x\ln x - \int 1\,dx\right]$$

$$= 2(x\ln x) - 2x + C$$

$u' = 1/x \qquad v = x$

$$\int \sin(2x)\, e^{\sin(x)}\,dx$$

$$= \int 2\sin(x)\cos(x)\, e^{\sin(x)}\,dx = 2\int w e^{w}\,dw = 2\left[we^{w} - \int 1\cdot e^{w}\,dw\right]$$

$w = \sin x$

$\frac{dw}{dx} = \cos x$

$dx = \frac{dw}{\cos x}$

$u = w \qquad v' = e^{w}$

$u' = 1 \qquad v = e^{w}$

$$= 2(we^{w} - e^{w}) + C$$

$$= 2\sin(x)e^{\sin x} - 2e^{\sin x} + C$$

$$\int_0^{\pi} x^3 \sin(x)\,dx$$

$u = x^3 \qquad v' = \sin x$

$u' = 3x^2 \qquad v = -\cos x$

$$= -x^3\cos(x)\Big|_0^{\pi} - \int_0^{\pi} 3x^2(-\cos(x))\,dx$$

$$= -x^3\cos(x)\Big|_0^{\pi} + 3\int_0^{\pi} x^2\cos(x)\,dx$$

$s = x^2 \qquad t' = \cos(x)$

$s' = 2x \qquad t = \sin(x)$

$$= -x^3\cos(x)\Big|_0^{\pi} + 3\left[x^2\sin(x)\Big|_0^{\pi} - \int_0^{\pi} 2x\sin x\,dx\right]$$

$$= -x^3\cos(x)\Big|_0^{\pi} + 3\left[x^2\sin(x)\Big|_0^{\pi} - 2\int_0^{\pi} x\sin(x)\,dx\right]$$

$a = x \qquad b' = \sin(x)$

$a' = 1 \qquad b = -\cos(x)$

$$= -x^3\cos(x)\Big|_0^{\pi} + 3\left[x^2\sin(x)\Big|_0^{\pi} - 2\left(-x\cos(x)\Big|_0^{\pi} - \int(-\cos(x))\,dx\right]\right.$$

$$= -x^3\cos(x)\Big|_0^{\pi} + 3\left[\cancel{x^2\sin(x)\Big|_0^{\pi}} - 2\left(-x\cos(x)\Big|_0^{\pi} + \cancel{\sin(x)\Big|_0^{\pi}}\right]\right.$$

$$= (-\pi^3(-1)) - (-\pi^3\cdot 1) + 6(\pi\cdot(-1) - \pi\cdot 1)$$

$$= 2\pi^3 - 12\pi$$

$\int e^x \cos(x)\,dx$ $\quad u = e^x \quad v' = \cos(x) \quad u' = e^x \quad v = \sin(x)$

$= e^x \sin(x) - \int e^x \sin(x)\,dx \quad s = \sin(x) \quad t' = e^x \quad s' = \cos(x) \quad t = e^x$

↳ undid what we just did! don't

$\cancel{=} e^x \sin(x) - [e^x \sin(x) - \int e^x \cos(x)\,dx]$

these cancel

$= e^x \sin(x) - \int e^x \sin(x)\,dx \quad s = e^x \quad t' = \sin(x) \quad s' = e^x \quad t = -\cos(x)$

$= e^x \sin(x) - [-e^x \cos(x) - \int e^x(-\cos(x))\,dx]$

$= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx$

So, $2\left[\int e^x \cos(x)\,dx\right] = e^x \sin(x) + e^x \cos(x)$

$\Rightarrow \int e^x \cos(x)\,dx = \frac{e^x \sin(x) + e^x \cos(x)}{2} + C$

Area Between Curves

$$A = \int_{-2}^{3} \left(-\frac{x^4}{8} + \frac{x^2}{2} + x + 1\right) - \left(-\frac{x}{8} - \frac{5}{4}\right) dx$$

$$= \int_{-2}^{3} \frac{-x^4}{8} + \frac{x^2}{2} + \frac{9x}{8} + \frac{9}{4}\, dx$$

$$= \frac{-x^5}{40} + \frac{x^3}{6} + \frac{9x^2}{16} + \frac{9x}{4}\, dx \Bigg|_{-2}^{3}$$

$$= \left(\frac{-243}{40} + \frac{27}{6} + \frac{81}{16} + \frac{27}{4}\right) - \left(\frac{32}{40} - \frac{8}{6} - \frac{36}{16} - \frac{18}{4}\right)$$

$$= -\frac{275}{40} + \frac{35}{6} + \frac{127}{16} + \frac{45}{4}$$

$$= \frac{625}{48} \approx 13.0208$$

$f(x) = \frac{-x^4}{8} + \frac{x^2}{2} + x + 1$ Intersects: solve $f(x) = g(x)$

$g(x) = \frac{-x}{8} - \frac{5}{4}$

$$-\frac{x^4}{8} + \frac{x^2}{2} + x + 1 = \frac{-x}{8} - \frac{5}{4}$$

$$-x^4 + 4x^2 + 8x + 8 = -x - 10$$

$$-x^4 + 4x^2 + 9x + 18 = 0$$

If $f(x)$ has a rational root (0), then it is a divisor of 18

divisors of 18: $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

↳ -2 works!

$\Rightarrow x - (-2)$ is a factor of $-x^4 + 4x^2 + 9x + 18$

$$\begin{array}{r l}
 & -x^3 + 2x^2 + 9 \\
x+2 & \overline{\big)\, -x^4 + 0x^3 + 4x^2 + 9x + 18} = 0 \\
 & -(-x^4 - 2x^3) \\
\hline
 & 2x^3 + 4x^2 \\
 & -(2x^3 + 4x^2) \\
\hline
 & 0 + 9x + 18 \\
 & -(9x + 18) \\
\hline
 & 0
\end{array}$$

$$\Rightarrow (x+2)(-x^3 + 2x^2 + 9) = f(x)$$

Rational sol to $(-x^3 + 2x^2 + 9) = 0$ are divisors of 9:

$\pm 1, \pm 3, \pm 9$

$x = 3$ is a sol, $x - 3$ is a factor

$$\begin{array}{r l}
 & -x^2 - x - 3 \\
x-3 & \overline{\big)\, -x^3 + 2x^2 + 0x + 9} \\
 & -(-x^3 + 3x^2) \\
\hline
 & -x^2 \\
 & -(-x^2 + 3x) \\
\hline
 & 0 - 3x + 9 \\
 & -(3x + 9) \\
\hline
 & 0
\end{array}$$

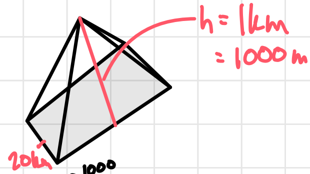
$$\Rightarrow -x^3 + 2x^2 + 9 = (x-3)(-x^2 - x - 3)$$

sol's to $-x^2 - x - 3 = 0$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-1)(-3)}}{2(-1)}$$

$$= \frac{1 \pm \sqrt{-11}}{-2}$$

Volume



$A(x) = ?$

$A(x) = s^2 = \frac{x^2}{25}$

$s = \frac{1000}{200} = \frac{x}{s}$

$\Rightarrow x = 5s$

$\Rightarrow s = \frac{x}{5}$

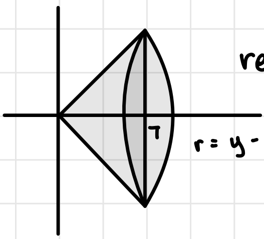
$V_{pyr} = \int_0^{1000} A(x)\,dx$

$= \int \frac{x^2}{25}\,dx = \frac{x^3}{3} \cdot \frac{1}{25} \Big|_0^{1000}$

$= \frac{x^3}{75} \Big|_0^{1000} = 1.\bar{3} \times 10^7$

Volumes of Solids of Revolution

Revolve a 2-D region on axis to get a solid w circular symmetry

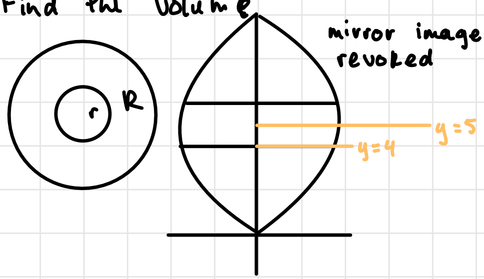


revolve the region betw $y = x$ and $y = 0$
$0 \le x \le 7$, about $y = 0$

$r = y - 0 = x$

$\int_0^7 \pi r^2\,dx = \int_0^7 \pi x^2\,dx = \frac{\pi x^3}{3}\Big|_0^7 = \frac{343\pi}{3}$

Revolve the region between $y = x^2$ and $y = 4$ about the line $y = 5$
Find the volume



$r = 5 - y$

$R = 5 - x^2$

$A = \pi R^2 - \pi r^2$

$\int_{-2}^{2} \pi (R^2 - r^2)\,dx = \pi \int_{-2}^{2} \left[(5 - x^2)^2 - 1^2\right] dx$

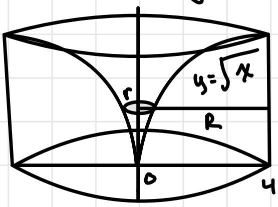
$V = \pi \int_{-2}^{2} \left[(5 - x^2)^2 - 1^2\right] dx = \pi \int_{-2}^{2} (25 - 10x^2 + x^4 - 1)\,dx$

$= \pi \left(24x - \frac{10x^3}{3} + \frac{x^5}{5}\right)\Big|_{-2}^{2} = \pi \left[\left(48 - \frac{80}{3} + \frac{32}{5}\right) - \left(-48 + \frac{80}{3} - \frac{32}{5}\right)\right]$

$= \pi \left(96 - \frac{160}{3} + \frac{64}{5}\right)$

More solids of Revolution & their Volumes

Revolve the Region between $y = \sqrt{x}$ and $y = 0$ for $0 \le x \le 4$ about the y-axis. What is the volume of the resulting solid?



The cross-sections are stacked parallel to y, so we'll use y as the basic variable. So, we express x as a function of y first.

$y = \sqrt{x} \quad 0 \le x \le 4$

$\Updownarrow$

$x = y^2 \quad 0 \le y \le 2$

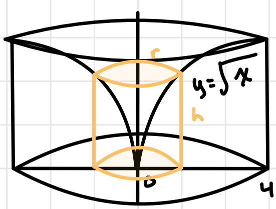
The region is between $\left.\begin{matrix} x = y^2 \\ \& \ x = 4 \end{matrix}\right) \ 0 \le y \le 2$

$r = x - 0 = y^2 - 0 = y^2$

$R = 4$

$$V = \int_0^2 \pi (R^2 - r^2)\, dy$$
$$= \pi \int_0^2 (4^2 - {y^2}^2)\, dy$$

$$= \pi \left(16y - \frac{y^5}{5}\right)\Big|_0^2$$
$$= \pi \left(32 - \frac{32}{5}\right) - 0$$
$$= \pi \cdot 32 \cdot \frac{4}{5} = \frac{128\pi}{5}$$



What is the volume of the resulting solid?

Area of open cylinder of radius r & height h is $2\pi r h$

For $0 \le x \le 4$ the cylinder at x has area $2\pi r h$ where $h = y - 0 = \sqrt{x}$

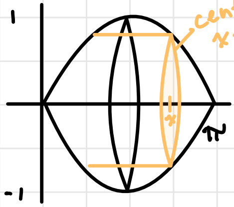
$r = x - 0 = x$

$$V = \int_0^4 \text{area of c.s}\; dx$$
$$= \int_0^4 2\pi r h\, dx$$
$$= \int_0^4 2\pi x \sqrt{x}\, dx$$
$$= 2\pi \int_0^4 x^{3/2}\, dx$$

$$= \frac{2\pi x^{5/2}}{5/2}\Big|_0^4$$
$$= \frac{4\pi}{5} \cdot x^{5/2}\Big|_0^4$$
$$= \frac{4\pi}{5} \cdot 4^{5/2} - 0$$
$$= \frac{4\pi}{5} \cdot (\sqrt{4})^5 \rightarrow = \frac{4\pi}{5} \cdot 2^5$$
$$= \frac{4\pi}{5} (1024)^{1/2}$$
$$= \frac{4\pi}{5} \cdot 32 = \frac{128\pi}{5}$$

Consider the region between $y=\sin(x)$ $(0 \le x \le \pi)$ and the x-axis.

a) Revolve this region about the x-axis



$R=\sin(x)$

$r=0$

Using cylinders

$r=\sin(x)$

$h=?$

$x=\arcsin(y)$

Need to use y:

$h=y-0 \quad (0 \le y \le 1)$

$r=(\pi-\arcsin(y))-\arcsin(y))$

$=\pi-2\arcsin(y)$

$$\int_0^1 2\pi r h\,dy = \int_0^1 2\pi y(\pi - 2\arcsin(y))\,dy$$

$\therefore$ use disks, it is easier!

Using disks

$$\int_0^{\pi} \pi(R^2-r^2)\,dx$$

$$= \pi \int_0^{\pi} \sin^2(x)\,dx$$

$\cos(2x)=\cos^2(x)-\sin^2(x)$

$=1-2\sin^2(x)$

$\sin^2(x) = \frac{1-\cos(2x)}{2}$

$$= \pi \int_0^{\pi} \frac{1-\cos(2x)}{2}\,dx$$

$u=2x$

$du=2dx$

$\frac{1}{2}du=dx$

x	u
0	0
π	2π

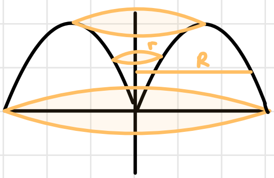
$$= \frac{\pi}{4}\int_0^{2\pi} 1-\cos(u)\,du$$

$$= \frac{\pi}{4}(u-\sin(u))\Big|_0^{2\pi}$$

$$= \frac{\pi}{4}(2\pi - \cancel{\sin(2\pi)}) - \cancel{\frac{\pi}{4}(0-\sin(0))}$$

$$= \frac{\pi^2}{2}$$

b) Revolve this region about the y-axis



$r=x=\arcsin(y)$

$R=\pi-x=\pi-\arcsin(y)$

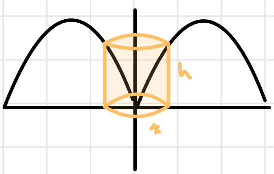
$0 \le y \le 1$

Using disks

$$V=\int_0^1 \pi(R^2-r^2)\,dy$$

$$=\pi[(\pi-\arcsin(y))^2-(\arcsin(y))^2]\,dy$$

... too hard, use cylinders



Using cylinders

$$V=\int_0^{\pi} 2\pi r h\,dx = \int_0^{\pi} 2\pi x\sin(x)\,dx$$

$u=x \quad v'=\sin(x)$

$u'=1 \quad v=-\cos(x)$

$$=2\pi\left[-x\cos(x)\Big|_0^{\pi} - \int_0^{\pi} 1\cdot(-\cos(x))\,dx\right]$$

$$=2\pi\left[-\pi\cos(\pi) - 0\cdot\cos(0) + \sin(x)\Big|_0^{\pi}\right.$$

$$=2\pi^2$$

Exam Structure

Page 1: 1. choose 4/6 derivatives (5 each)
2. Choose 4/6 integrals (5 each)
3. Choose 4/6 various (eps-delt, lims...) (5 each)
4. Qualitative analysis / curve sketching (12)

Page 2. 5. Choose 2/3 (14 each)

Total = 100pts.

Bonus: 1. creative writing (1)
2. math... look into coleridge?