An Alternate Version of the ε–δ Definition of Limits

The usual ε–δ definition of limits,

> Definition. $\lim_{x \to a} f(x) = L$ exactly when for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any x with $|x - a| < \delta$ we are guaranteed to have $|f(x) - L| < \varepsilon$ as well.

is pretty hard to wrap your head around the first time or three for most people. Given the intricate logical structure of the definition – "for all ... there is ... for any ... if ... , then ... " – this is probably only to be expected.

Here is less common definition, equivalent to the one above, that is easier for some people* to understand and use than the standard one. Even for those for whom the standard version of the ε–δ definition of limits comes more easily, it may be useful to parse the following definition to achieve a better understanding of the standard definition. The alternate definition recasts the somewhat confusing logical structure of the standard definition in terms of a game:

> Alternate Definition. The *limit game* for $f(x)$ at $x = a$ with target L is a three-move game played between two players A and B as follows:
>
> 1. A moves first, picking a small number $\varepsilon > 0$.
> 2. B moves second, picking another small number $\delta > 0$.
> 3. A moves third, picking an x that is within δ of a, *i.e.* $a - \delta < x < a + \delta$.
>
> To determine the winner, we evaluate $f(x)$. If it is within ε of the target L, *i.e.* $L - \varepsilon < f(x) < L + \varepsilon$, then player B wins; if not, then player A wins.
>
> With this idea in hand, $\lim_{x \to a} f(x) = L$ means that player B has a winning strategy in the limit game for $f(x)$ at $x = a$ with target L; that is, if B plays it right, B will win no matter what A tries to do. (Within the rules ... :-) Conversely, $\lim_{x \to a} f(x) \neq L$ means that player A is the one with a winning strategy in the limit game for $f(x)$ at $x = a$ with target L.

The game definition of limits isn't really better or worse that the usual $\varepsilon - \delta$ definition, but each is easier for some people to understand, and the exercise in trying it both ways usually helps in understanding what is really going on here.

Example 1. We will use the alternate definition of limit to verify that $\lim_{x \to 3} (3x - 4) = 5$.

To verify that $\lim_{x \to 3} (3x - 4) = 5$ according to the alternate version of the ε–δ definition we need to show that player B has a winning strategy. To develop one, let's review how the game works in this case:

1. A chooses an $\varepsilon > 0$.
2. B responds by choosing a $\delta > 0$.
3. A then picks an x with $3 - \delta < x < 3 + \delta$.

- B wins if $5 - \varepsilon < 3x - 4 < 5 + \varepsilon$.

Note that since B has no control over As moves, a winning strategy for B must work no matter what $\varepsilon > 0$ is chosen by A on move 1 and what x is chosen by A on move

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* A not entirely insignificant minority, anyway.

3 (subject to the requirement that $3 - \delta < x < 3 + \delta$, of course). As with applying the standard ε–δ definition, we will reverse-engineer the δ that B should play from the winning condition:

$$\begin{aligned} 5 - \varepsilon < 3x - 4 < 5 + \varepsilon &\iff -\varepsilon < 3x - 4 - 5 < \varepsilon \\ &\iff -\varepsilon < 3x - 9 < \varepsilon \\ &\iff -\varepsilon < 3(x - 3) < \varepsilon \\ &\iff -\frac{\varepsilon}{3} < x - 3 < \frac{\varepsilon}{3} \\ &\iff 3 - \frac{\varepsilon}{3} < x < 3 + \frac{\varepsilon}{3} \end{aligned}$$

Comparing the last part to the requirement that A play an x with $3 - \delta < x < 3 + \delta$ and observing that every step in the above chain of reasoning is reversible, we see that if B plays $\delta = \frac{\varepsilon}{3}$ in response to A playing $\varepsilon > 0$, then B wins: any x that A can play with $3 - \frac{\varepsilon}{3} < x < 3 + \frac{\varepsilon}{3}$ will result in having $5 - \varepsilon < 3x - 4 < 5 + \varepsilon$, which is the winning condition for B.

Thus B has a winning strategy – "play $\delta = \frac{\varepsilon}{3}$" – and so $\lim_{x\to 3}(3x - 4) = 5$ by the alternate version of the ε–δ definition of limits. □

Example 2. We will use the alternate definition of limit to verify that $\lim_{x\to 2} x^2 \neq 0$.

To verify that $\lim_{x\to 2} x^2 \neq 0$ according to the alternate version of the ε–δ definition we need to show that player A has a winning strategy. To develop one, let's review how the game works in this case:

1. A chooses an $\varepsilon > 0$.
2. B responds by choosing a $\delta > 0$.
3. A then picks an x with $2 - \delta < x < 2 + \delta$.

- A wins if $x^2 - 0 \leq 2 - \varepsilon$ or $x^2 - 0 \geq 2 + \varepsilon$.

Examining this sequence tells us that a winning strategy for A requires us to choose an $\varepsilon > 0$ such that no matter what $\delta > 0$ player B chooses, there is some way to choose an x with $2 - \delta < x < 2 + \delta$, and such that $x^2 \leq 2 - \varepsilon$ or $x^2 \geq 2 + \varepsilon$.

We choose an $\varepsilon > 0$ for A to play that is smaller than the difference between the alleged limit of 0 and what the function x^2 really achieves at $x = 2$, which is $2^2 = 4$. Since $0 < 1 < 4 - 0 = 4$, $\varepsilon = 1$ is a convenient choice.

Now suppose B plays $\delta > 0$. We will have A play $x = 2 + \frac{\delta}{2}$. (Why? This falls into the required range of x's, $2 - \delta < x < 2 + \delta$, and is to the right of $x = 2$. Since x^2 is increasing near $x = 2$, this puts the corresponding point on the graph even farther away from the alleged limit of 0.) This wins for A because

$$\left(2 + \frac{\delta}{2}\right)^2 - 0 = \left(2 + \frac{\delta}{2}\right)^2 = 2^2 + 2 \cdot \frac{\delta}{2} + \delta^2 = 4 + \delta + \delta^2 > 4 > 3 = 2 + 1 = 2 + \varepsilon,$$

as required.

Thus A has a winning strategy in the limit game: play $\varepsilon = 1$ on the first move, and then respond with $x = 2 + \dfrac{\delta}{2}$ after B plays any $\delta > 0$. It follows by the alternate version of the ε–δ definition of limits that $\lim_{x \to 2} x^2 \neq 0$. $\square$