A Few Useful Bits of Algebra and Trigonometry
Briefly Summarized

Difference of squares

If a is any constant, then $x^2 - a^2 = (x-a)(x+a)$. Since $a^2 \geq 0$ for any real number a, it follows that $x^2 - C = \left(x - \sqrt{C}\right)\left(x + \sqrt{C}\right)$ whenever $C \geq 0$. (Just take $C = a^2$ ...)

By contrast, this does not work for $x^2 + C = x - (-C)$ if $C > 0$, as this would require $\sqrt{-C}$ to be a real number.

The quadratic formula

The solutions of the equation $ax^2 + bx + c = 0$, where $p \neq 0$, are given by

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

These solutions are the *roots* of the quadratic ax^2+bx+c. Note that if the *discriminant* of the equation, the b^2-4ac inside the square root, is negative, then the equation $ax^2+bx+c = 0$ cannot have a solution x that is a real number.

If we set $r = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $s = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$, then $ax^2 + bx + c = a(x - r)(x-s)$, so the quadratic formula also gives us a handy way to factor quadratic expressions into linear factors when it is possible to do so. If the discriminant $b^2 - 4ac$ is negative, the quadratic has no real roots and it is impossible to factor the quadratic expression into linear factors with real coefficients, in which case the quadratic is said to be *irreducible*.

Completing the square

"Completing the square" on the quadratic $px^2 + qx + r$ works as follows:

$$\begin{aligned} px^2 + qx + r &= p\left[x^2 + \frac{q}{p}x + \frac{r}{p}\right] = p\left[\left(x + \frac{q}{2p}\right)^2 - \frac{q^2}{4p^2} + \frac{r}{p}\right] \\ &= p\left(x + \frac{q}{2p}\right)^2 + \left(r - \frac{q^2}{4p}\right) \end{aligned}$$

This has several uses, including proving the quadratic formula, simplifying the job of solving the equation $px^2+qx+r = 0$ if one did not wish to use the quadratic formula, and, when integrating, setting up substitutions like $u = x + \dfrac{q}{2p}$ to simplify integrands involving the quadratic expression $px^2 + qx + r$.

Roots and linear factors of polynomials

A *polynomial* in the variable x is a sum of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where each a_i is a real number. The largest integer $n \geq 0$ for which the coefficient $a_n \neq 0$ is the *degree* of the polynomial. (A small exception is the constant function $f(x) = 0$, which is a polynomial with all coefficients $= 0$, but is still considered to be of degree 0.)

Polynomials of degree 0 are said to be *constant*; of degree 1, *linear*; of degree 2, *quadratic*; of degree 3, *cubic*; of degree 4, *quartic*; and of degree 5, *quintic*. (One could go on, but people usually don't.*)

Some useful facts about polynomials:

- Every polynomial of degree > 0 can be written as a product of (powers of) linear factors – that is, polynomials of degree 1 – and/or irreducible quadratic factors – that is, polynomials of degree 2 that have no roots.
- A polynomial $p(x)$ has a real number a as a *root*, *i.e.* $p(a) = 0$, if and only if $x - a$ is a factor of $p(x)$, that is, $p(x) = (x - a)q(x)$ for some polynomial $q(x)$ of degree one less than the degree of $p(x)$.

A minimal set of trigonometric identities

- $\sin^2(x) + \cos^2(x) = 1$
  [Often used in the form $\cos^2(x) = 1 - \sin^2(x)$ or $\sin^2(x) = 1 - \cos^2(x)$.]
- $1 + \tan^2(x) = \sec^2(x)$
  [Sometimes used in the form $\sec^2(x) - 1 = \tan^2(x)$.]
- $\sin(2x) = 2\sin(x)\cos(x)$
- $$\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x)\\ &= 2\cos^2(x) - 1\\ &= 1 - 2\sin^2(x)\end{aligned}$$
  [Sometimes used in the form $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$ or $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$.]

It is also useful to keep in mind that:

- $\sin(x)$ and $\cos(x)$ are *periodic* with period 2π: for any real number x and any integer n, $\sin(x + 2n\pi) = \sin(x)$ and $\cos(x + 2n\pi) = \cos(x)$.
- $\sin(x)$ is an *odd* function, $\sin(-x) = -\sin(x)$ for all x, and $\cos(x)$ is an *even* function, $\cos(-x) = \cos(x)$ for all x.
- Phase shifts are fun: for any real number x, $\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$, $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$, $\sin(x \pm \pi) = -\sin(x)$, and $\cos(x \pm \pi) = -\cos(x)$.

While these are not exactly trigonometric identities, it is a good thing to remember that $-1 \leq \sin(x) \leq 1$ and $-1 \leq \cos(x) \leq 1$, *i.e.* $|\sin(x)| \leq 1$ and $|\cos(x)| \leq 1$, for all x.

* Why not? Possibly because if one followed the Latin-style numbering of quartic and quintic, the next two would be *sextic* and *septic*, respectively. :-)