Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals (Section C)
TRENT UNIVERSITY, Fall 2021

Solutions to Quiz #9
Wednesday, 24 November.

Compute each of the following three integrals.

1. $\displaystyle\int_0^{\pi/2} \frac{\sin(x)}{1+\cos^2(x)}\,dx$ *[1.5]*

SOLUTION. We will use the substitution $u = \cos(x)$, so $du = -\sin(x)\,dx$ and $\sin(x)\,dx = (-1)\,du$, and change the limits as we go along: $\begin{matrix} x & 0 & \pi/2 \\ u & 1 & 0 \end{matrix}$ We will also use the reversal property of definite integrals: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

$$\int_0^{\pi/2} \frac{\sin(x)}{1+\cos^2(x)}\,dx = \int_1^0 \frac{1}{1+u^2}\,(-1)\,du = -\int_1^0 \frac{1}{1+u^2}\,du = \int_0^1 \frac{1}{1+u^2}\,du$$
$$= \arctan(u)\Big|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4} \qquad \blacksquare$$

2. $\displaystyle\int_0^{\pi/2} \sin^2(x)\cos^3(x)\,dx$ *[1.5]*

SOLUTION. We will use the trigonometric identity $\cos^2(\theta) = 1 - \sin^2(\theta)$ and the substitution $w = \sin(x)$, so $dw = \cos(x)\,dx$, and change the limits as we go along: $\begin{matrix} x & 0 & \pi/2 \\ w & 0 & 1 \end{matrix}$

$$\begin{aligned}
\int_0^{\pi/2} \sin^2(x)\cos^3(x)\,dx &= \int_0^{\pi/2} \sin^2(x)\cos^2(x)\cos(x)\,dx \\
&= \int_0^{\pi/2} \sin^2(x)\left(1-\sin^2(x)\right)\cos(x)\,dx \\
&= \int_0^1 w^2\left(1-w^2\right)\,dw = \int_0^1 \left(w^2 - w^4\right)\,dw \\
&= \left(\frac{w^3}{3} - \frac{w^5}{5}\right)\Big|_0^1 = \left(\frac{1^3}{3} - \frac{1^5}{5}\right) - \left(\frac{0^3}{3} - \frac{0^5}{5}\right) \\
&= \frac{1}{3} - \frac{1}{5} - 0 = \frac{2}{15} \qquad \blacksquare
\end{aligned}$$

3. $\displaystyle\int \frac{e^{2x}}{1+e^x}\,dx$ *[2]*

SOLUTION. The key here is that $e^{2x} = (e^x)^2 = e^x e^x$. This will allow us to use the substitution $t = 1 + e^x$, so $dt = e^x\,dx$ and $e^x = t - 1$.

$$\int \frac{e^{2x}}{1+e^x}\,dx = \int \frac{e^x e^x}{1+e^x}\,dx = \int \frac{t-1}{t}\,dt = \int \left(1 - \frac{1}{t}\right)\,dt$$
$$= t - \ln(t) + C = 1 + e^x - \ln\left(1+e^x\right) + C$$

This could also have been done with a two-stage substitution: first $s = e^x$, so $ds = e^x\,dx$, and then $t = 1 + s$, so $dt = ds$. $\blacksquare$