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Integration by Parts III – Integrals of Powers of Trigonometric Functions

$$\int \sec^3(x)\,dx$$

$u = \sec(x)$ $\qquad v' = \sec^2(x)$

$u' = \sec(x)\tan(x)$ $\qquad v = \tan(x)$

$$= \sec(x)\tan(x) - \int \sec(x)\tan(x)\tan(x)\,dx$$

$$= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx$$

$1 + \tan^2(x) = \sec^2(x)$

$\sec^2(x) - 1 = \tan^2(x)$

$$= \sec(x)\tan(x) - \int \sec(x)\left(\sec^2(x) - 1\right)dx$$

$$= \sec(x)\tan(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx$$

$$\Rightarrow 2\int \sec^3(x)\,dx = \sec(x)\tan(x) + \int \sec(x)\,dx$$

$$\Rightarrow \int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\,dx$$

Aside: $\int \sec(x)\,dx$ We will complicate the integrand to make a substitution possible.

$$= \int \sec(x) \cdot \frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx = \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx$$

$$w = \sec(x)+\tan(x)$$

$$dw = (\sec(x)\tan(x) + \sec^2(x))\,dx$$

$$= \int \frac{1}{w}\,dw$$

$$= \ln(w) + C = \ln(\sec(x)+\tan(x)) + C$$

$$\therefore \int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln(\sec(x)+\tan(x)) + C$$

In general, if $n \geq 2$, then

$$\int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx,$$

by a very similar process. (Do it as an exercise.)

There are such "reduction" formulas for all the usual trig functions. For example, if $n \geq 2$, ③

$$\boxed{\int \sin^n(x)\,dx}$$

$u = \sin^{n-1}(x)$ $\qquad v' = \sin(x)$

$u' = (n-1)\sin^{n-2}(x)\cos(x)$ $\qquad v = -\cos(x)$

$$= -\sin^{n-1}(x)\cos(x) - \int (n-1)\sin^{n-2}(x)\cos(x)(-\cos(x))\,dx$$

$$= -\sin^{n-1}(x)\cos(x) + \int (n-1)\sin^{n-2}(x)\cos^2(x)\,dx$$

$\cos^2(x) + \sin^2(x) = 1$

$\cos^2(x) = 1 - \sin^2(x)$

$$= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\left(1 - \sin^2(x)\right)dx$$

$$= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx - (n-1)\boxed{\int \sin^n(x)\,dx}$$

$$\Rightarrow n\int \sin^n(x)\,dx = -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx$$

$$\Rightarrow \boxed{\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx}$$

The other basic reduction formulas are:

$$(n \geq 2) \int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

$$(n \geq 2) \int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \overset{?}{\cancel{\frac{n-2}{n-1}}}\int \tan^{n-2}(x)\,dx$$

Do this as an exercise to check what the constant ought to be.

It ought to be 1.

There are other – and much more complex – reduction formulas for mixed powers of sin(x) & cos(x) & for sec(x) and tan(x), as well as reduction formulas for csc(x) and cot(x).