Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
Trent University, Fall 2020

Solutions to Assignment #2
Continuity and Differentiability
Due on Friday, 9 October.

1 Verify that $f(x) = |x|$ is continuous but not differentiable at $x = 0$. *[3]*

Solution. It is easy to check that $f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$ is continuous at $x = 0$. To do so, one needs to check that $\lim_{x \to 0} |x| = 0 = |0|$. One *could* use an ε–δ argument to show that, but it is easier to check that the one-sided limits at $x = 0$ both exists and are both equal to 0:

$$\begin{aligned} \lim_{x \to 0^-} |x| &= \lim_{x \to 0^-} (-x) = -\lim_{x \to 0^-} x = -0 = 0 = |0| \\ \lim_{x \to 0^+} |x| &= \lim_{x \to 0^+} x = 0 = |0| \end{aligned}$$

Since the one-sided limits from both directions exist and equal $0 = |0|$, $f(x) = |x|$ is continuous at $x = 0$.

For $f(x) = |x|$ to be differentiable at $x = 0$, the limit required by the definition of the derivative, namely

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h},$$

must exist. Since the corresponding one-sided derivatives,

$$\begin{aligned} \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0^-} \frac{|h|}{h} \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1 \\ \text{and} \quad \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0^+} \frac{|h|}{h} \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1, \end{aligned}$$

do not agree, $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}$, does not exist and so $f(x) = |x|$ is not differentiable at $x = 0$. $\square$

Consider the function $g(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$.

2. Verify that $g(x)$ is continuous at 0 and explain why $g(x)$ is continuous for all $x \neq 0$. *[2]*

SOLUTION. To verify that $g(x)$ is continuous at $x = 0$, we need to check that $\lim_{x \to 0} g(x) = g(0) = 0$. As $x \to 0$, $x \neq 0$, so $\lim_{x \to 0} g(x) = \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$. This last cannot simply be evaluated by plugging in $x = 0$ because dividing by 0 is ~~strictly forbidden~~ undefined. We will use the Squeeze Theorem to work around this problem.

Observe that $x^2 \geq 0$ for all x and that $-1 \leq \sin(t) \leq 1$ for all t. It follows that $-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$ for all $x \neq 0$. Since both $\lim_{x \to 0} \left(-x^2\right) = -0^2 = 0$ and $\lim_{x \to 0} x^2 = 0^2 = 0$, it follows by the Squeeze Theorem that $\lim_{x \to 0} g(x) = \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$ too. Since $g(0) = 0$ by the definition of $g(x)$, this means that $g(x)$ is continuous at $x = 0$.

For all $x \neq 0$ we have $g(x) = x^2 \sin\left(\frac{1}{x}\right)$, so $g(x)$ is continuous at all such x because it is the composition and product of functions which are continuous (and also differentiable) wherever they are defined. $\square$

3. Verify that $g(x)$ is differentiable at 0 and explain why $g(x)$ is differentiable for all $x \neq 0$. *[2]*

SOLUTION. To verify that $g(x)$ is differentiable at $x = 0$, we need to check that

$$g'(0) = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{g(h) - 0}{h} = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right)$$

exists, *i.e.* works out to a real number. Since $-1 \leq \sin(t) \leq 1$ for all t and $h \to 0$, $\lim_{h \to 0} h \sin\left(\frac{1}{h}\right)$ ought to equal 0. We'll use the Squeeze Theorem again in order to verify this.

Since $-|h| \leq h \sin\left(\frac{1}{h}\right) \leq |h|$ for all $h \neq 0$ and $\lim_{h \to 0} (-|h|) = -0 = 0 = \lim_{h \to 0} |h|$, it follows by the Squeeze Theorem that $\lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0$. By the limit definition of the derivative, this means that $g'(0)$ is defined and $= 0$.

For all $x \neq 0$ we can actually compute the derivative using the usual rules:

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left[x^2 \sin\left(\frac{1}{x}\right)\right] = \left[\frac{d}{dx} x^2\right] \sin\left(\frac{1}{x}\right) + x^2 \left[\frac{d}{dx} \sin\left(\frac{1}{x}\right)\right] \\ &= 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right) \cdot \frac{d}{dx}\left(\frac{1}{x}\right) = 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) \\ &= 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \end{aligned}$$

It follows, in particular, that $g(x)$ is differentiable (and hence continuous) for all $x \neq 0$. $\square$

4. Work out $g'(x)$ for all x and determine for which values of x it is continuous and/or differentiable at x. *[3]*

Hint: Recall that if a function is differentiable at some point, then it must be continuous at that point. (It follows that if it fails to be continuous at some point, it can't be differentiable there either.) This can save you a bit of time in question **4**.

SOLUTION. This question asks us to repeat what we did for questions **1** and **2** for $g'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ instead of $g(x)$. We will use the hint to avoid duplicating more work than we really need to.

First, observe that for all $x \neq 0$, we can compute the derivative of $g'(x)$ using the usual rules:

$$\begin{aligned}
g''(x) &= \frac{d}{dx}g'(x) = \frac{d}{dx}\left[2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right] \\
&= \left[\frac{d}{dx}2x\right]\sin\left(\frac{1}{x}\right) + 2x\left[\frac{d}{dx}\sin\left(\frac{1}{x}\right)\right] - \frac{d}{dx}\cos\left(\frac{1}{x}\right) \\
&= 2\sin\left(\frac{1}{x}\right) + 2x\cos\left(\frac{1}{x}\right)\cdot\frac{d}{dx}\left(\frac{1}{x}\right) - \left[-\sin\left(\frac{1}{x}\right)\right]\cdot\frac{d}{dx}\left(\frac{1}{x}\right) \\
&= 2\sin\left(\frac{1}{x}\right) + 2x\cos\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right) + \sin\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right) \\
&= 2\sin\left(\frac{1}{x}\right) - \frac{2}{x}\cos\left(\frac{1}{x}\right) - \frac{1}{x^2}\sin\left(\frac{1}{x}\right) \\
&= \left(1 - \frac{1}{x^2}\right)\sin\left(\frac{1}{x}\right) - \frac{2}{x}\cos\left(\frac{1}{x}\right)
\end{aligned}$$

Since we can do so, and the resulting expression makes sense for all $x \neq 0$, $g'(x)$ is differentiable, and hence also continuous, for all $x \neq 0$.

Second, $g'(x)$ is not continuous at $x = 0$. To be sure, $g'(0) = 0$ is defined, by our solution to **3** above, but it turns out that $\lim_{x\to 0} g'(x)$ is undefined, much less is equal to 0. Note that the first part of $g'(x)$, namely $2x\sin\left(\frac{1}{x}\right)$, does have a limit of 0 as $x \to 0$: since $-1 \leq \sin(t) \leq 1$ for all t, it follows that $-2|x| \leq 2x\sin\left(\frac{1}{x}\right) \leq 2|x|$. Since $\lim_{x\to 0}(-2|x|) = 0 = \lim_{x\to 0} 2|x|$, it follows by the Squeeze Theorem that $\lim_{x\to 0} 2x\sin\left(\frac{1}{x}\right) = 0$ too. However, this means that for $\lim_{x\to 0} g'(x)$ to exist (and $= 0$), $\lim_{x\to 0}\cos\left(\frac{1}{x}\right)$ would have to exist (and $= 0$), but it does not. (I would be happy to accept an informal argument for this, but I'll provide a more careful one below.)

Consider, for example, the numbers $x_n = \frac{1}{n\pi}$, where n is an integer ≥ 1. These head off to 0 as n increases, so if $\lim_{x\to 0}\cos\left(\frac{1}{x}\right)$ existed, it would have to equal $\lim_{n\to\infty}\cos\left(\frac{1}{x_n}\right) =$

$\lim_{n\to\infty} \cos(n\pi)$. Suppose this last limit did exist. Since when n is odd, $\cos\left(\frac{1}{x_n}\right) = \cos(n\pi) = -1$, and when n is even, $\cos\left(\frac{1}{x_n}\right) = \cos(n\pi) = 1$, it follows that $\lim_{n\to\infty} \cos\left(\frac{1}{x_n}\right) = \lim_{n\to\infty} \cos(n\pi)$ would have to be both -1 and 1, which is impossible. Thus $\lim_{n\to\infty} \cos\left(\frac{1}{x_n}\right) = \lim_{n\to\infty} \cos(n\pi)$, and thus $\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$ cannot exist.

It follows that $\lim_{x\to 0} g'(x)$ does not exist, so $g'(x)$ is not continuous, and hence also not differentiable, at $x = 0$. ■