Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Fall 2020

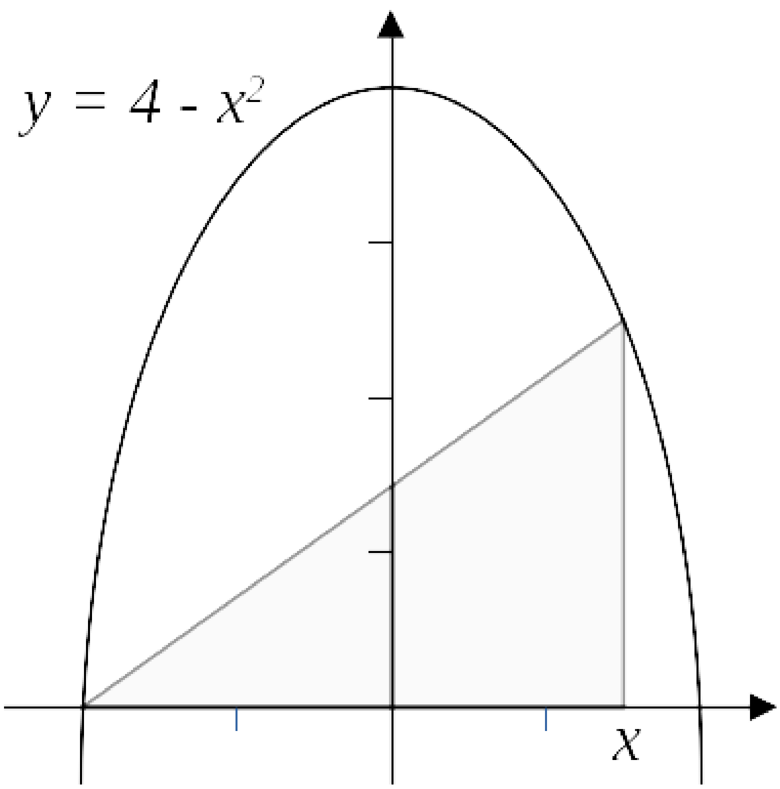
Solutions to Quiz #7
Tuesday, 10 November .

Consider the following setup:

> A right triangle has one end of the hypotenuse at the point $(-2, 0)$ and the other end on a point of the parabola $y = 4 - x^2$ with $y \geq 0$. The base of the triangle lies along the x-axis and the altitude of the triangle is parallel to the y-axis.

1. Sketch the setup. *[1]*

SOLUTION. Here's a fairly crude sketch of the setup:



□

2. What is the maximum area of such a triangle? *[4]*

SOLUTION. A triangle in this this setup has base $x-(-2) = x+2$ and height $y-0 = 4-x^2$, so it has area $A(x) = \frac{1}{2}\cdot\text{base}\cdot\text{height} = \frac{1}{2}(x+2)\left(4 - x^2\right)$. Also, for the end of the hypotenuse on the parabola to have $y = 4 - x^2 \geq 0$, we have to have $-2 \leq x \leq 2$. Thus our task is to maximize $A(x) = \frac{1}{2}(x+2)\left(4 - x^2\right)$ on the interval $[-2, 2]$, which we will do by comparing the value of $A(x)$ at the endpoints with its value at any critical points inside the interval.

At the endpoints we have

$$A(-2) = \frac{1}{2}(-2+2)\left(4-(-2)^2\right) = \frac{1}{2}\cdot 0 \cdot 0 = 0$$
$$\text{and } A(2) = \frac{1}{2}(2+2)\left(4-2^2\right) = \frac{1}{2}\cdot 4 \cdot 0 = 0\,.$$

To find the critical points we need to find where $A'(x) = 0$ or is undefined. To compute $A'(x)$ it's convenient to first expand our expression for $A(x)$:

$$A(x) = \frac{1}{2}(x+2)\left(4-x^2\right) = \frac{1}{2}\left(4x - x^3 + 8 - 2x^2\right) = \frac{1}{2}\left(-x^3 - 2x^2 + 4x - 8\right)$$

Taking the derivative of this gives:

$$A'(x) = \frac{d}{dx}\frac{1}{2}\left(-x^3 - 2x^2 + 4x - 8\right) = \frac{1}{2}\left(-3x^2 - 4x + 4 - 0\right) = -\frac{1}{2}\left(3x^2 + 4x - 4\right)$$

Note that $A'(x)$, like $A(x)$, is defined and continuous for all x, so we only need to look for points where $A'(x) = 0$.

$$A'(x) = 0 \iff 3x^2 + 4x - 4 = 0 \iff x = \frac{-4 \pm \sqrt{4^2 - 4\cdot 3\cdot(-4)}}{2\cdot 3} = \frac{-4 \pm \sqrt{16+48}}{6}$$
$$= \frac{-4 \pm \sqrt{64}}{6} = \frac{-4 \pm 8}{6} = \frac{-2 \pm 4}{3}\,,$$

Thus $A'(x) = 0$ exactly when $x = \dfrac{-2+4}{3} = \dfrac{2}{3}$ or $x = \dfrac{-2-4}{3} = -2$, both of which are in the interval $[-2, 2]$. We have already worked out that $A(-2) = 0$ because $x = -2$ is an endpoint of the interval, so it remains to compute $A\left(\frac{2}{3}\right)$:

$$A\left(\frac{2}{3}\right) = \frac{1}{2}\left(\frac{2}{3}+2\right)\left(4-\left(\frac{2}{3}\right)^2\right) = \frac{1}{2}\cdot\frac{8}{3}\cdot\left(4-\frac{4}{9}\right) = \frac{4}{3}\cdot\frac{32}{9} = \frac{128}{27}$$

Since $A\left(\dfrac{2}{3}\right) = \dfrac{128}{27} > 0 = A(-2) = A(2)$, it follows that the maximum area of a triangle in the given setup is $\dfrac{128}{27} \approx 4.74$. ■