Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Fall 2020

Solutions to Quiz #4
Tuesday, 13 October .

Consider the function $g(x) = \dfrac{x^2}{1-x^2}$ and do *both* of the following problems:

1. Find all the vertical and horizontal asymptotes, if any, of $y = g(x)$. *[3]*

SOLUTION. We look for vertical asymptotes first. $g(x) = \dfrac{x^2}{1-x^2}$ is continuous (and differentiable) everywhere it is defined, so it can only have vertical asymptotes at points where its definition does not make sense, *i.e.* when the denominator is 0. $1 - x^2 = 0$ exactly when $x = \pm 1$, so we check on either side of each of these points for a vertical asymptote.

Note that when $x < -1$ or $x > 1$, $x^2 > 1$, so $1 - x^2 < 0$, and when $-1 < x < 1$, $x^2 < 1$, so $1 - x^2 > 0$. Thus $1 - x^2 \to 0^-$ when $x \to -1^1$ or $x \to +1^+$, and $1 - x^2 \to 0^+$ when $x \to -1^+$ or $x \to +1^-$. Obviously, $x^2 \to 1$ as $x \to \pm 1$. It now follows that:

$$\lim_{x\to -1^-} \frac{x^2}{1-x^2} \begin{matrix} \to 1 \\ \to 0^- \end{matrix} = -\infty$$
$$\lim_{x\to -1^+} \frac{x^2}{1-x^2} \begin{matrix} \to 1 \\ \to 0^+ \end{matrix} = +\infty$$
$$\lim_{x\to +1^-} \frac{x^2}{1-x^2} \begin{matrix} \to 1 \\ \to 0^+ \end{matrix} = +\infty$$
$$\lim_{x\to +1^+} \frac{x^2}{1-x^2} \begin{matrix} \to 1 \\ \to 0^- \end{matrix} = -\infty$$

Hence $g(x)$ has vertical asymptotes at both $x = -1$ and $x = +1$, heading to $-\infty$ when approaching -1 from the left or $+1$ from the right, and heading to $+\infty$ when approaching -1 from the right or $+1$ from the left.

We now check for horizontal asymptotes.

$$\lim_{x\to -\infty} \frac{x^2}{1-x^2} = \lim_{x\to -\infty} \frac{x^2}{1-x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to -\infty} \frac{1}{\frac{1}{x^2} - 1} = \frac{1}{0-1} = -1^-$$
$$\lim_{x\to +\infty} \frac{x^2}{1-x^2} = \lim_{x\to +\infty} \frac{x^2}{1-x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to +\infty} \frac{1}{\frac{1}{x^2} - 1} = \frac{1}{0-1} = -1^-$$

Note that $\frac{1}{x^2} > 0$, so $\frac{1}{\frac{1}{x^2}-1} < -1$ as $x \to \pm\infty$, so both limits approach -1 from below. Also, since $x^2 \to +\infty$ and $1 - x^2 \to -\infty$ as $x \to \pm\infty$, so we could have used l'Hôpital's Rule to compute the limits above.

Hence $g(x)$ has the horizontal asymptote $y = -1$ in both directions, which it approaches from below in both directions. □

2. Find all the local maxima and minima, if any, of $y = g(x)$. *[2]*

SOLUTION. We first compute the derivative of $g(x)$ with the help of the quotient rule:

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(\frac{x^2}{1-x^2}\right) = \frac{\left[\frac{d}{dx}x^2\right]\left(1-x^2\right) - x^2\left[\frac{d}{dx}\left(1-x^2\right)\right]}{\left(1-x^2\right)^2} \\ &= \frac{[2x]\left(1-x^2\right) - x^2[-2x]}{\left(1-x^2\right)^2} = \frac{2x-2x^3+2x^3}{\left(1-x^2\right)^2} = \frac{2x}{\left(1-x^2\right)^2} \end{aligned}$$

Note that $g'(x)$ is undefined exactly where $g(x)$ is undefined, namely at $x = \pm 1$. (We know from solving **1** above that $g(x)$ has vertical asymptotes at these points.)

Next, we find all the critical points of $g(x)$:

$$g'(x) = \frac{2x}{\left(1-x^2\right)^2} = 0 \iff 2x = 0 \iff x = 0$$

Thus our only candidate for a local maximum or minimum is $x = 0$. Note that $g(0) = 0$.

Finally, we check to see if the critical point we found is a local maximum, a local minimum, or neither. Here are two ways to do so.

Method 1 – testing points: We compare the value of $g(x)$ at $x = 0$, $g(0) = 0$, to nearby points on either side of it. It's important that we pick points that are closer to $x = 0$ than any vertical asymptotes on that side of $x = 0$, since the behaviour of a function can shift radically on the other side of an asymptote. As the asymptotes are at $x = \pm 1$, we test $g(x)$ at $x = \pm\frac{1}{2}$:

$$\begin{aligned} g\left(-\frac{1}{2}\right) &= \frac{\left(-\frac{1}{2}\right)^2}{1-\left(-\frac{1}{2}\right)^2} = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{4}\cdot\frac{4}{3} = \frac{1}{3} \\ g\left(+\frac{1}{2}\right) &= \frac{\left(+\frac{1}{2}\right)^2}{1-\left(+\frac{1}{2}\right)^2} = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{4}\cdot\frac{4}{3} = \frac{1}{3} \end{aligned}$$

Since $g\left(-\frac{1}{2}\right) = g\left(+\frac{1}{2}\right) = \frac{1}{3} > 0 = g(0)$, it follows that $x = 0$ is a local minimum of $g(x)$, at which point $g(0) = 0$.

Method 2 – intervals of increase and decrease: We analyze $g'(x) = \dfrac{2x}{\left(1-x^2\right)^2}$ to see where it is positive, and thus $g(x)$ is increasing, and where it is negative, and thus $g(x)$ is decreasing. Since the denominator, $\left(1-x^2\right)^2$, of $g'(x)$ is a square, it is always ≥ 0, while the numerator, $2x$, is positive or negative (or 0) exactly when x is positive or negatove (or 0). It follows that $g'(x)$ is negative for all $x < 0$ where it is defined (*i.e.* except at $x = -1$, where both $g(x)$ and $g'(x)$ are undefined) and positive for all $x > 0$ where it is defined (*i.e.* except at $x = +1$, where both $g(x)$ and $g'(x)$ are undefined). It follows that $g(x)$ is decreasing when $x < 0$ and increasing when $x > 0$, so $x = 0$ is a local minimum of $g(x)$, at which point $g(0) = 0$.

The reasoning above would often be summarized in a table, into which we'll throw in part of what we learned from question **1** too:

x	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \infty)$
$g'(x)$	$-$	undef	$-$	0	$+$	undef	$+$
$g(x)$	$\downarrow$	VA	$\downarrow$	local min	$\uparrow$	VA	$\uparrow$

We'll usually take this approach when we do full-blown qualitative analysis/curve sketching. ■