Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Fall 2020

Quiz #3
Tuesday, 6 October .

Do *both* of the following problems:

1. A curve is defined by the equation $x = \arctan(1-y) + \sqrt{y(2-y)}$. Find the slope of the tangent line to this curve at the point $(1,1)$. *[2.5]*

SOLUTION. One could *try* to solve for y in terms of x and then differentiate, but good luck with that ... That leaves implicit differentiation as the only reasonable option to find $\frac{dy}{dx}$.

$$\begin{aligned}
x &= \arctan(1-y) + \sqrt{y(2-y)} \\
\Rightarrow 1 = \frac{d}{dx}x &= \frac{d}{dx}\left[\arctan(1-y) + \sqrt{y(2-y)}\right] = \frac{d}{dx}\arctan(1-y) + \frac{d}{dx}\sqrt{y(2-y)} \\
&= \frac{1}{1+(1-y)^2} \cdot \frac{d}{dx}(1-y) + \frac{1}{2\sqrt{y(2-y)}} \cdot \frac{d}{dx}\left(y(2-y)\right) \\
&= \frac{1}{1+(1-y)^2}\left(0 - \frac{dy}{dx}\right) + \frac{1}{2\sqrt{y(2-y)}}\left[\left(\frac{dy}{dx}\right)(2-y) + y\left(\frac{d}{dx}(2-y)\right)\right] \\
&= \frac{-1}{1+(1-y)^2} \cdot \frac{dy}{dx} + \frac{1}{2\sqrt{y(2-y)}}\left[\left(\frac{dy}{dx}\right)(2-y) + y\left(0 - \frac{dy}{dx}\right)\right] \\
&= \frac{-1}{1+(1-y)^2} \cdot \frac{dy}{dx} + \frac{1}{2\sqrt{y(2-y)}} \cdot (2-2y)\frac{dy}{dx} \\
&= \left[\frac{-1}{1+(1-y)^2} + \frac{1-y}{\sqrt{y(2-y)}}\right] \cdot \frac{dy}{dx} \\
\Rightarrow \frac{dy}{dx} &= \frac{1}{\dfrac{-1}{1+(1-y)^2} + \dfrac{1-y}{\sqrt{y(2-y)}}}
\end{aligned}$$

It follows that the slope of the tangent line at the point $(x,y) = (1,1)$ on the curve is:

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)} = \frac{1}{\dfrac{-1}{1+(1-1)^2} + \dfrac{1-1}{\sqrt{1(2-1)}}} = \frac{1}{\dfrac{-1}{1+0^2} + \dfrac{0}{\sqrt{1 \cdot 1}}} = \frac{1}{-1+0} = -1 \quad \square$$

2. Compute $\lim_{x \to -\infty} x^3 e^x$. *[2.5]*

SOLUTION. Note that as $x \to -\infty$, $x^3 \to -\infty$ too, but $e^x \to 0$. We will rewrite the given limit so that we can apply l'Hôpital's Rule using the fact that $e^{-x} = \dfrac{1}{e^x} \to \infty$ if $e^{-x} \to 0$.

(Recall that $e^t > 0$ for all t.) We will use the fact that $\frac{d}{dt}e^{-t} = e^{-t} \cdot \frac{d}{dt}(-t) = -e^{-t}$ several times over.

$$\begin{aligned}
\lim_{x\to-\infty} x^3 e^x &= \lim_{x\to-\infty} \frac{x^3 \quad \to -\infty}{e^{-x} \to +\infty} = \lim_{x\to-\infty} \frac{\frac{d}{dx}x^3}{\frac{d}{dx}e^{-x}} \quad \text{by l'Hôpital's Rule} \\
&= \lim_{x\to-\infty} \frac{3x^2 \quad \to +\infty}{-e^{-x} \to -\infty} = \lim_{x\to-\infty} \frac{\frac{d}{dx}3x^2}{\frac{d}{dx}\left(-e^{-x}\right)} \quad \text{by l'Hôpital's Rule} \\
&= \lim_{x\to-\infty} \frac{6x \qquad \to -\infty}{-\left(-e^{-x}\right) \to +\infty} = \lim_{x\to-\infty} \frac{\frac{d}{dx}6x}{\frac{d}{dx}e^{-x}} \quad \text{by l'Hôpital's Rule} \\
&= \lim_{x\to-\infty} \frac{6 \quad \to 6}{-e^{-x} \to -\infty} = 0 \qquad \blacksquare
\end{aligned}$$