Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Fall 2020

Quiz #2
Tuesday, 29 September.

Do *both* of the following problems:

1. Use the limit definition of the derivative to work out the derivative of $f(x) = \dfrac{1}{1+x^2}$. *[2.5]*

SOLUTION. Here we go, in entirely too much detail:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{1+(x+h)^2} - \frac{1}{1+x^2}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{1+(x+h)^2} - \frac{1}{1+x^2}\right)$$
$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{x^2}{(1+x^2)\left(1+(x+h)^2\right)} - \frac{(x+h)^2}{(1+x^2)\left(1+(x+h)^2\right)}\right)$$
$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{x^2 - \left(x^2+2hx+h^2\right)}{(1+x^2)\left(1+(x+h)^2\right)}\right) = \lim_{h\to 0} \frac{-2hx-h^2}{h\,(1+x^2)\left(1+(x+h)^2\right)}$$
$$= \lim_{h\to 0} \frac{h(-2x-h)}{h\,(1+x^2)\left(1+(x+h)^2\right)} = \lim_{h\to 0} \frac{-2x-h}{(1+x^2)\left(1+(x+h)^2\right)}$$
$$= \frac{-2x-0}{(1+x^2)\left(1+(x+0)^2\right)} = \frac{-2x}{(1+x^2)(1+x^2)} = \frac{-2x}{(1+x^2)^2} \qquad \square$$

2. Compute[†] the derivative of $g(x) = (e^x)^{\cos^2(x)}\left(e^{\sin^2(x)}\right)^x\left(e^{\sec^2(x)}\right)^{-x}(e^x)^{\tan^2(x)}$. For full credit, make your solution as efficient as possible. *[2.5]*

SOLUTION. It is probably not a good idea to just start taking the derivative of the given expression for $g(x)$. It works, *if* you are patient and careful enough, but it is much better to simplify that expression first, with the help of the properties of exponents and trigonometric functions:

$$g(x) = (e^x)^{\cos^2(x)}\left(e^{\sin^2(x)}\right)^x\left(e^{\sec^2(x)}\right)^{-x}(e^x)^{\tan^2(x)}$$
$$= e^{x\cos^2(x)}e^{x\sin^2(x)}e^{-x\sec^2(x)}e^{x\tan^2(x)} = e^{x\cos^2(x)+x\sin^2(x)-x\sec^2(x)+x\tan^2(x)}$$
$$= e^{x\left(\cos^2(x)+\sin^2(x)-\sec^2(x)+\tan^2(x)\right)} = e^{x(1-1)} = e^{x\cdot 0} = e^0 = 1$$

Recall that $\cos^2(x)+\sin^2(x) = 1$ and $\tan^2(x)+1 = \sec^2(x)$ (so $-\sec^2(x)+\tan^2(x) = -1$). It follows that $g'(x) = \dfrac{d}{dx}1 = 0$. ■

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† Using algebra and the practical rules for computing derivatives. You do *not* have to verify you are correct using the limit definition of the derivative. (Again, unless you're a mathochist ... :-)