**Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals**
TRENT UNIVERSITY, Fall 2020

**Solutions to Quiz #11 – The last, at last!**
*Tuesday, 8 December .*

Do *one* (1) of problems **1** or **2**.

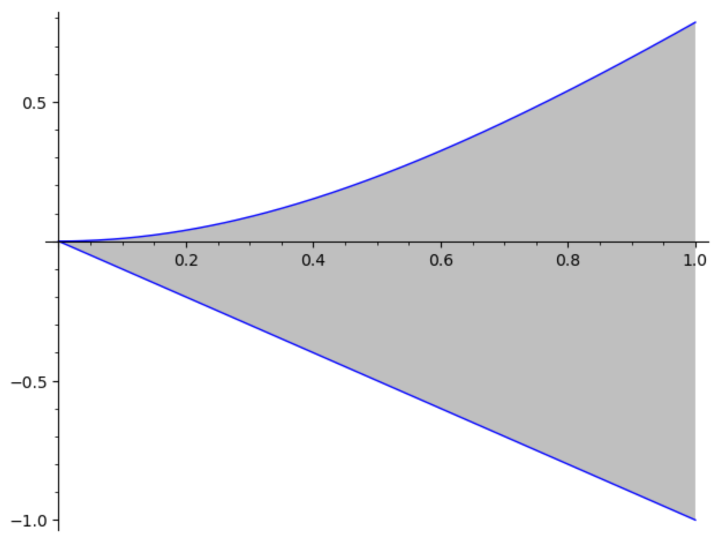
**1.** Find the area of the region between $y = x\arctan(x)$ and $y = -x$, for $0 \le x \le 1$. *[5]*

SOLUTION. With a region between two curves, it's often a good idea to sketch the region, or at least the curves bounding it, to get an idea of how to set up the integral. In particular, one needs to determine which curve is above the other on the given interval.

Cheating a bit, I plotted the region in `SageMath`:

```
sage: plot(x*arctan(x),0,1,fill=True) + plot(-x,0,1,fill=True)
```

This gave the following output:




It's pretty clear from the plot that when $0 \le x \le 1$, we have $y = x\arctan(x)$ above $y = -x$.

One could also sort this out by observing that $x$ and $\arctan(x)$ are both $\ge 0$ for $x \ge 0$, and hence so is $x\arctan(x)$, while $-x \le 0$ when $x \ge 0$. It then follows that $x\arctan(x)$ is above $-x$ when $0 \le x \le 1$.

Either way, this lets us set up and compute the integral giving the area of the region:

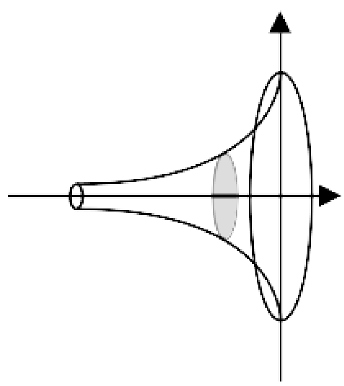
$$\begin{aligned}
\text{Area} &= \int_0^1 (\text{upper} - \text{lower})\, dx = \int_0^1 (x\arctan(x) - (-x))\, dx \\
&= \int_0^1 (x\arctan(x) + x)\, dx = \int_0^1 x\arctan(x)\, dx + \int_0^1 x\, dx
\end{aligned}$$

We will use parts on the integral of $x\arctan(x)$, with $u = \arctan(x)$ and $v' = x$, so $u' = \dfrac{1}{1+x^2}$ and $v = \dfrac{x^2}{2}$.

$$\begin{aligned}
&= \frac{x^2}{2}\arctan(x)\Big|_0^1 - \int_0^1 \frac{1}{1+x^2}\cdot\frac{x^2}{2}\, dx + \frac{x^2}{2}\Big|_0^1 \\
&= \frac{1^2}{2}\arctan(1) - \frac{0^2}{2}\arctan(0) - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}\, dx + \frac{1^2}{2} - \frac{0^2}{2} \\
&= \frac{1}{2}\cdot\frac{\pi}{4} - 0 - \frac{1}{2}\int_0^1 \frac{1+x^2-1}{1+x^2}\, dx + \frac{1}{2} - 0 \\
&= \frac{\pi}{8} + \frac{1}{2} - \frac{1}{2}\left[\int_0^1 \frac{1+x^2}{1+x^2}\, dx - \int_0^1 \frac{1}{1+x^2}\, dx\right] \\
&= \frac{\pi}{8} + \frac{1}{2} - \frac{1}{2}\int_0^1 1\, dx + \frac{1}{2}\int_0^1 \frac{1}{1+x^2}\, dx = \frac{\pi}{8} + \frac{1}{2} - \frac{x}{2}\Big|_0^1 + \frac{1}{2}\arctan(x)\Big|_0^1 \\
&= \frac{\pi}{8} + \frac{1}{2} - \left[\frac{1}{2} - \frac{0}{2}\right] + \left[\frac{1}{2}\arctan(1) - \frac{1}{2}\arctan(0)\right] \\
&= \frac{\pi}{8} + \frac{1}{2} - \frac{1}{2} + \left[\frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\cdot 0\right] = \frac{\pi}{8} + \frac{\pi}{8} = \frac{\pi}{4} \qquad \square
\end{aligned}$$

**2.** Find the volume of the solid obtained by revolving the region between $y = e^x$ and the $x$-axis, for $-\ln(5) \le x \le 0$, about the $x$-axis. *[5]*

SOLUTION. Here's a crude sketch of the vaguely trumpet-shaped solid:



We will find its volume using the disk/washer method. The vertical cross-section of the solid at $x$ is a disk with radius $r = e^x - 0 = e^x$, and hence with area $A(x) = \pi r^2 = \pi\left(e^x\right)^2 = \pi e^{2x}$. It follows that the volume fo the solid is:

$$\text{Volume} = \int_{-\ln(5)}^{0} A(x)\,dx = \int_{-\ln(5)}^{0} \pi r^2\,dx = \int_{-\ln(5)}^{0} \pi e^{2x}\,dx$$

We will use the substitution $w = 2x$, so $dw = 2\,dx$ and $dx = \dfrac{1}{2}\,dw$, changing the limits as we go along: $\begin{array}{ccc} x & -\ln(5) & 0 \\ w & -2\ln(5) & 0 \end{array}$

$$= \int_{-2\ln(5)}^{0} \pi e^{w}\left(\frac{1}{2}\right)dw = \frac{\pi}{2}\int_{-2\ln(5)}^{0} e^{w}\,dw = \frac{\pi}{2}e^{w}\Big|_{-2\ln(5)}^{0}$$

$$= \frac{\pi}{2}e^{0} - \frac{\pi}{2}e^{-2\ln(5)} = \frac{\pi}{2}\cdot 1 - \frac{\pi}{2}\cdot\frac{1}{e^{2\ln(5)}} = \frac{\pi}{2} - \frac{\pi}{2}\cdot\frac{1}{\left(e^{\ln(5)}\right)^2}$$

$$= \frac{\pi}{2} - \frac{\pi}{2}\cdot\frac{1}{5^2} = \frac{\pi}{2} - \frac{\pi}{2}\cdot\frac{1}{25} = \frac{25\pi}{50} - \frac{\pi}{50} = \frac{24\pi}{50} = \frac{12\pi}{25} \qquad \blacksquare$$