Integration by Parts

Substitution is running the Chain Rule backwards...
Integration by Parts is running the Product Rule for derivatives backwards with a little rearrangement:

Product Rule: $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)\cdot g'(x)$

If $u = f(x)$ & $v = g(x)$, then we can write this as $(u\cdot v)' = u'v + u\cdot v'$

$$\Rightarrow u\cdot v' = (u\cdot v)' - u'\cdot v$$

Integration by Parts:

$$\int u\cdot v'\,dx = \int [(u\cdot v)' - u'\cdot v]\,dx$$
$$= \int (u\cdot v)'\,dx - \int u'\cdot v\,dx$$
$$= uv - \int u'\cdot v\,dx$$

The formula is: $\boxed{\int u\cdot v'\,dx = uv - \int u'\cdot v\,dx}$

Example: $\int x e^x \, dx$

We want to think of xe^x as $u \cdot v'$, so we have to divide up xe^x into u and v'.

The idea is usually to do so that $\int u' \cdot v \, dx$ is easier to handle.

Try $u = x$ & $v' = e^x$,

then $u' = 1$ & $v = e^x$,

So
$$\underbrace{\int x e^x \, dx}_{u \cdot v'} = \underbrace{x e^x}_{u v} - \underbrace{\int 1 \cdot e^x \, dx}_{u' \cdot v} = xe^x - e^x + C$$
$$= (x-1)e^x + C$$

If we had tried $u = e^x$ & $v' = x$ instead, we'd have

$u' = e^x$ & $v = \frac{x^2}{2}$,

so
$$\underbrace{\int x e^x \, dx}_{v' \cdot u} = \underbrace{\frac{x^2}{2} e^x}_{v \cdot u} - \underbrace{\int \frac{x^2}{2} e^x \, dx}_{v \cdot u'}$$

This is harder than what we started with.

This is why ⟶ matters.

Another example: $\displaystyle\int_0^{\pi/2} x^2\cos(x)\,dx$

[Having dissimilar functions multiplied in the integrand is often a clue to try Parts.]

Try $u = x^2 \quad v' = \cos(x)$ and

$u' = 2x \quad v = \sin(x)$,

then
$$\int_0^{\pi/2} \underset{u\cdot v'}{x^2\cos(x)}\,dx = \underset{u\cdot v}{x^2\sin(x)}\Big|_0^{\pi/2} - \int_0^{\pi/2} \underset{u'\cdot v}{2x\sin(x)}\,dx$$

$$= x^2\sin(x)\Big|_0^{\pi/2} - 2\int_0^{\pi/2} \underset{s\cdot t'}{x\sin(x)}\,dx$$

Do parts again...

$s = x$ & $t' = \sin(x)$

$s' = 1$ & $t = -\cos(x)$

$$= x^2\sin(x)\Big|_0^{\pi/2} + 2\left[-x\cos(x)\Big|_0^{\pi/2} + \int_0^{\pi/2} 1\cdot(-\cos(x))\,dx\right]$$

$$= x^2\sin(x)\Big|_0^{\pi/2} + 2\left[x\cos(x)\Big|_0^{\pi/2} - \sin(x)\Big|_0^{\pi/2}\right]$$

$$= \left(\frac{\pi^2}{4}\cdot 1 - 0^2\cdot 0\right) + 2\left[\left(\frac{\pi}{2}\cdot 0 - 0\cdot 1\right) - (1-0)\right]$$

$$= \boxed{\frac{\pi^2}{4} - 2}$$

Example: $\int \ln(x)\,dx$ Use Parts? How?

Cheat: $= \int \underset{v'}{1} \cdot \underset{u}{\ln(x)}\,dx$ $u = \ln(x)$ & $v' = 1$

so $u' = \frac{1}{x}$ & $v = x$

$= \underset{v \cdot u}{x \ln(x)} - \int \frac{1}{x} \cdot x\,dx = x\ln(x) - \int 1\,dx = \boxed{x\ln(x) - x + C}$

$= x(\ln(x) - 1) + C$

This same trick can be used to compute

$\int \arctan(x)\,dx$ $u = \arctan(x)$ $v' = 1$

$u' = \frac{1}{1+x^2}$ $v = x$

Substitution: $w = 1+x^2$

so $dw = 2x\,dx$

& $x\,dx = \frac{1}{2}dw$

$= x\arctan(x) - \int \frac{1}{1+x^2} \cdot x\,dx$

$= x\arctan(x) - \int \frac{1}{w} \cdot \frac{1}{2}dw$

$= x\arctan(x) - \frac{1}{2}\int \frac{1}{w}dw = x\arctan(x) - \frac{1}{2}\ln(w) + C$

$= \boxed{x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C}$

One more: $\int_1^e x^2 \ln(x)\,dx$

$u = x^2 \qquad v' = \ln(x)$

$u' = 2x \qquad v = ?\ x\ln(x) - x$

$= x^2(x\ln(x) - x)\Big|_1^e - \int_1^e 2x(x\ln(x) - x)\,dx$

$= (x^3\ln(x) - x^3)\Big|_1^e - 2\int_1^e (x^2\ln(x) - x^2)\,dx$

$= (e^3\ln(e) - e^3) - (1^3\ln(1) - 1^3) - 2\int_1^e x^2\ln(x)\,dx + 2\int_1^e x^2\,dx$

$= (e^3 \cdot 1 - e^3) - (0 - 1) - 2\int_1^e x^2\ln(x)\,dx + 2\frac{x^3}{3}\Big|_1^e$

$= 1 - 2\int_1^e x^2\ln(x)\,dx + \frac{2}{3}e^3 - \frac{2}{3}\cdot 1^3 = \left[\frac{2}{3}e^3 + \frac{1}{3}\right] - 2\int_1^e x^2\ln(x)\,dx$

So we have $\boxed{\int_1^e x^2\ln(x)\,dx} = \left[\frac{2}{3}e^3 + \frac{1}{3}\right] - 2\boxed{\int_1^e x^2\ln(x)\,dx}$

Solve the eqn. for $\boxed{\cdots}$

$\Rightarrow \quad 3\boxed{\int_1^e x^2\ln(x)\,dx} = \frac{2}{3}e^3 + \frac{1}{3}$

$\Rightarrow \quad \boxed{\int_1^e x^2\ln(x)\,dx} = \frac{2}{9}e^3 + \frac{1}{9}$

We made a poor choice, but a lot of algebra made it work anyway.

A better choice would have been

$$\int_1^e x^2 \ln(x)\,dx \qquad u = \ln(x) \quad v' = x^2$$

$$u' = \frac{1}{x} \quad v' = \frac{x^3}{3}$$

$$= \frac{x^3}{3}\ln(x)\Big|_1^e - \int_1^e \frac{1}{x}\cdot\frac{x^{2}}{3}\,dx$$

$$= \left[\frac{e^3}{3}\ln(e) - \frac{1^3}{3}\ln(1)\right] - \frac{1}{3}\int_1^e x^2\,dx$$

$$= \frac{e^3}{3}\cdot 1 - \frac{1}{3}\cdot 0 - \frac{1}{3}\cdot\frac{x^3}{3}\Big|_1^e \qquad = \frac{3e^3}{9}$$

$$= \frac{e^3}{3} - \left[\frac{1}{9}e^3 - \frac{1}{9}\cdot 1^3\right] = \frac{e^3}{3} - \left[\frac{e^3}{9} - \frac{1}{9}\right]$$

$$= \frac{2e^3}{9} + \frac{1}{9}$$