Examples of Substitution

A quick recap of integration so far:

Fundamental Thm. of Calculus (I)

If $F'(x) = f(x)$ on $[a,b]$, and $f(x)$ is defined and integrable on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

[$F(x)$ is an antiderivative of $f(x)$.]

weighted area between $y = f(x)$ & the x-axis, with area above the axis being added and area below being subtracted.

We often use the notation $\int f(x)\,dx$ to represent or refer to a generic antiderivative of $f(x)$. So if $F(x)$ is any particular antiderivative of $f(x)$, then

$$\int f(x)\,dx = F(x) + C,$$

where C is a generic constant, since any two particular antiderivatives of $f(x)$ differ by some constant.

Rules for computing integrals:

(1) $\int x^n \, dx = \begin{cases} \frac{x^{n+1}}{n+1} & \text{if } n \neq -1 \\ \ln(x) & \text{if } n = -1 \end{cases}$

(Power Rule)

(2) $\int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx$ [Sum Rule]

(3) $\int c f(x) \, dx = c \int f(x) \, dx$ [Multiplication by Constants Rule]

(4) $\int f(g(x)) \cdot g'(x) \, dx = \int f(u) \, du$ where $\underbrace{u = g(x)}_{\text{Substitution}}$

(Substitution Rule) Usually, when you work this out you substitute back.

For definite integrals, we can change the limits as we go along (and then don't have to substitute back).

ie $\int_a^b f(g(x)) \cdot g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du$. $\left[\begin{array}{l} u = g(x) \\ du = g'(x) dx \end{array} \right]$

Examples of using the Substitution Rule:

0° $\displaystyle\int_0^{\pi/4} \underbrace{\tan^3(\theta)}_{u^3}\, \underbrace{\sec^2(\theta)\,d\theta}_{du}$

Observe that $\frac{d}{d\theta}\tan(\theta) = \sec^2(\theta)$.

Try $u = \tan(\theta)$, so $du = \sec^2(\theta)\,d\theta$

Change the limits as we go along:

θ	$u = \tan(\theta)$
0	$\tan(0) = 0$
$\frac{\pi}{4}$	$\tan(\frac{\pi}{4}) = 1$

$$= \int_0^1 u^3\,du = \left(\frac{u^4}{4} + C\right)\Bigg|_0^1$$

$$= \left(\frac{1^4}{4} + C\right) - \left(\frac{0^4}{4} + C\right)$$

$$= \frac{1}{4} + \cancel{C} - 0 - \cancel{C} = \boxed{\frac{1}{4}}$$

1° $\displaystyle\int x\sqrt{5x+1}\,dx$

Try to simplify the integral, by substituting $u = 5x+1$. $\left[x = \frac{u-1}{5}\right]$

Then $du = 5dx$, so $dx = \frac{1}{5}du$

$$= \int \frac{u-1}{5}\underbrace{\sqrt{u}}_{u^{1/2}} \cdot \frac{1}{5}\,du$$

$$= \frac{1}{25}\int (u-1)\sqrt{u}\,du = \frac{1}{25}\int \left(u^{3/2} - u^{1/2}\right)du = \frac{1}{25}\left(\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right) + C$$

$$= \frac{1}{25}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C = \frac{1}{25}\left(\frac{2}{5}(5x+1)^{5/2} - \frac{2}{3}(5x+1)^{3/2}\right) + C$$

(4)

2° $\displaystyle\int_0^{\pi/4} \sec^4(\theta)\,d\theta$

No obvious substitution, so we'll try to rewrite the integral to make one possible. Use the identity $1+\tan^2(\theta)=\sec^2(\theta)$ and the fact that $\frac{d}{d\theta}\tan(\theta)=\sec^2(\theta)$

$$=\int_0^{\pi/4} \sec^2(\theta)\cdot\sec^2(\theta)\,d\theta$$

$$=\int_0^{\pi/4} (1+\tan^2(\theta))\sec^2(\theta)\,d\theta$$

Substitute $u=\tan(\theta)$, so $du=\sec^2(\theta)\,d\theta$, and change the limits:

θ	u
0	0
$\frac{\pi}{4}$	1

$$=\int_0^1 (1+u^2)\,du=\left(u+\frac{u^3}{3}\right)\Big|_0^1$$

$$=\left(1+\frac{1^3}{3}\right)-\left(0+\frac{0^3}{3}\right)=1+\frac{1}{3}-0=\frac{4}{3}.$$

3° $\displaystyle\int_0^1 \frac{x+1}{x^2+1}\,dx$

We can't substitute for x^2 or x^2+1, since we don't have an isolated multiple of x available. Note that $\frac{d}{dx}(x^2)=\frac{d}{dx}(x^2+1)=2x$.

but...

$$=\int_0^1 \frac{x}{x^2+1}\,dx+\int_0^1 \frac{1}{x^2+1}\,dx=\int_1^2 \frac{1}{u}\cdot\frac{1}{2}\,du+\arctan(x)\Big|_0^1$$

$u=x^2+1$
$du=2x\,dx$
$x\,dx=\frac{1}{2}du$

x	u
0	1
1	2

$$=\frac{1}{2}\ln(u)\Big|_1^2+\arctan(1)-\arctan(0)$$

$$\to=\frac{1}{2}\ln(2)-\underbrace{\frac{1}{2}\ln(1)}_{=0}+\frac{\pi}{4}-0$$

$$=\frac{1}{2}\ln(2)+\frac{\pi}{4}.$$

(5)

4° $\int \sin(x)\cos^2(x)\,dx$

Hard way is to substitute

$u = \sin(x)$, so $du = \cos(x)\,dx$

and $\cos(x) = \sqrt{1-\sin^2(x)} = \sqrt{1-u^2}$.

(since $\sin^2(x)+\cos^2(x)=1$) (& $\cos^2(x) = 1-u^2$)

$$\underset{\downarrow}{\sin(x)}\ \underset{\downarrow}{\cos(x)}\ \underset{\swarrow}{\cos(x)dx}$$

$$= \int u\sqrt{1-u^2}\,du$$

Now substitute again: $w = 1-u^2$,

so $dw = -2u\,du \Rightarrow u\,du = \left(-\frac{1}{2}\right)dw$

$$= \int \sqrt{w}\cdot\left(-\frac{1}{2}\right)dw$$

$$= -\frac{1}{2}\int w^{1/2}\,dw$$

$$= -\frac{1}{2}\cdot\frac{w^{3/2}}{3/2} + C = -\frac{1}{3}w^{3/2} + C = -\frac{1}{3}(1-u^2)^{3/2} + C$$

$$= -\frac{1}{3}\left(\cos^2(x)\right)^{3/2} + C = -\frac{1}{3}\cos^3(x) + C$$

Easier way is to substitute $t = \cos(x)$, so $dt = -\sin(x)\,dx$

and $(-1)dt = \sin(x)\,dx$

$$\int \sin(x)\cos^2(x)\,dx = \int t^2(-1)\,dt = -\frac{t^3}{3} + C$$

$$= -\frac{1}{3}\cos^3(x) + C.$$

5° $\displaystyle\int \frac{1+e^x}{1-e^x}\,dx$

We'll try to simplify by substituting for e^x, even though we don't have an isolated e^x to help us, since $\frac{d}{dx}e^x = e^x$.

$u = e^x \qquad du = e^x\,dx \qquad$ so $dx = \frac{1}{e^x}\,du = \frac{1}{u}\,du$

Can't mix variables so write as

$$= \int \frac{1+u}{1-u}\cdot\frac{1}{u}\,du$$

& now what? Algebraic trickery:

$$\frac{2}{1-u} + \frac{1}{u} = \frac{2u + 1-u}{(1-u)u} = \frac{1+u}{(1-u)u}$$

$$= \int \left(\frac{2}{1-u} + \frac{1}{u}\right)du = 2\int \frac{1}{1-u}\,du + \int \frac{1}{u}\,du$$

Substitute $w = 1-u$, so $dw = (-1)\,du$, so $du = (-1)\,dw$.

$$= 2\int \frac{1}{w}(-1)\,dw + \ln(u) = -2\ln(w) + \ln(u) + C$$

[Put in the generic constant once the integral sign is gone. Only for indefinite integrals.]

$$= -2\ln(1-u) + \ln(u) + C$$

$$= -2\ln(1-e^x) + \ln(e^x) + C$$

$$= \ln(e^x) - \ln\left((1-e^x)^2\right) + C = \ln\left(\frac{e^x}{(1-e^x)^2}\right) + C.$$