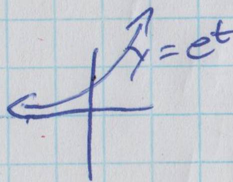
Asymptotes II – Some more examples

1° $f(x) = xe^{-x^2}$ is defined and continuous everywhere, so it has no vertical asymptotes

We check for horizontal asymptotes:

$$\lim_{x\to\infty} \underbrace{x}_{\to\infty}\, \underbrace{e^{\overset{\to -\infty}{-x^2}}}_{\to 0} = \lim_{x\to\infty} \frac{x^{\to\infty}}{e^{x^2}_{\to\infty}} \overset{\text{l'Hôpital's Rule}}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^{x^2}}$$

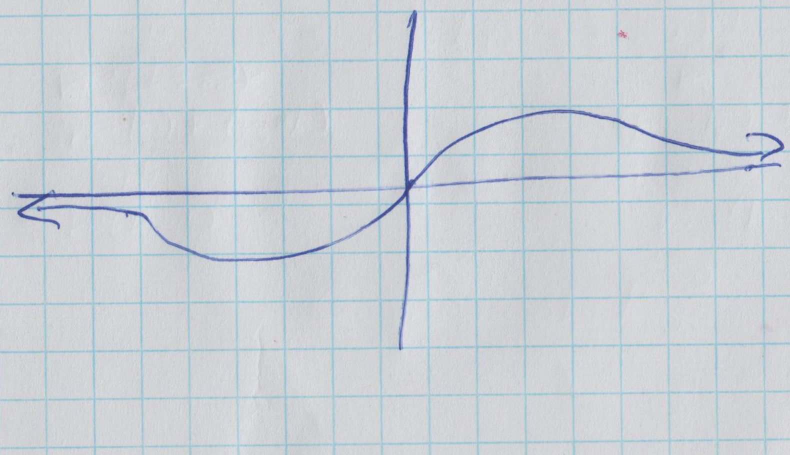


$$= \lim_{x\to\infty} \frac{1}{e^{x^2}\cdot\frac{d}{dx}(x^2)} = \lim_{x\to\infty} \frac{1^{\to 1}}{e^{x^2}\cdot 2x_{\to\infty}} = 0^+$$

$$\lim_{x\to-\infty} \underbrace{x}_{\to -\infty}\, \underbrace{e^{-x^2}}_{\to 0} = \lim_{x\to-\infty} \frac{x^{\to -\infty}}{e^{x^2}_{\to\infty}} = \lim_{x\to-\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^{x^2}} = \ldots$$

$$= \lim_{x\to-\infty} \frac{1^{\to 1}}{2xe^{x^2}_{\to -\infty}} = 0^-$$

$f(0) = 0$



∴ We have a H.A. of $y = 0$ in both directions.

2°

$g(x) = e^{-1/x^2}$ is defined and continuous everywhere except at $x=0$, where it is undefined.

Vertical asymptotes?

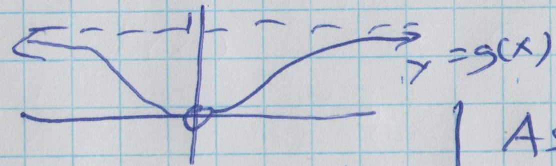
$$\lim_{x\to 0^-} e^{-1/x^2} = 0^+$$

$$\lim_{x\to 0^+} e^{-1/x^2} = 0^+$$

$\therefore$ $g(x)$ has no vertical asymptotes

As $x \to 0^+$,
then $x^2 \to 0^+$,
so $\frac{1}{x^2} \to +\infty$,
so $\frac{-1}{x^2} \to -\infty$,
so $e^{-1/x^2} \to 0^+$.

As $x \to 0^-$,
then $x^2 \to 0^+$,
so $\frac{1}{x^2} \to +\infty$,
so $\frac{-1}{x^2} \to -\infty$,
so $e^{-1/x^2} \to 0^+$.



Horizontal asymptotes?

$$\lim_{x\to -\infty} e^{-1/x^2} = \lim_{x\to -\infty} \frac{1}{e^{1/x^2}} = \frac{1}{1} = 1$$

As $x \to -\infty$,
$x^2 \to +\infty$,
$\frac{1}{x^2} \to 0^+$,
$e^{1/x^2} \to 1 = e^0$.

$$\lim_{x\to +\infty} e^{-1/x^2} = \lim_{x\to +\infty} \frac{1}{e^{1/x^2}} = \frac{1}{1} = 1$$

As $x \to +\infty$,
$x^2 \to +\infty$,
$\frac{1}{x^2} \to 0^+$,
$e^{1/x^2} \to 1 = e^0$

So this function has HA of $y=1$ in both directions.

3° $h(x) = \dfrac{1-x^2}{1+x^2}$ Since $1+x^2 \geq 1 > 0$, so this is defined and continuous for all x. Hence it has no V.A.

Horizontal asymptotes?

$$\lim_{x\to-\infty} \frac{1-x^2}{1+x^2} = \lim_{x\to-\infty} \frac{1-x^2}{1+x^2}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{\overbrace{\frac{1}{x^2}}^{\to 0}-1}{\underbrace{\frac{1}{x^2}}_{\to 0}+1} = \frac{0-1}{0+1} = -1$$

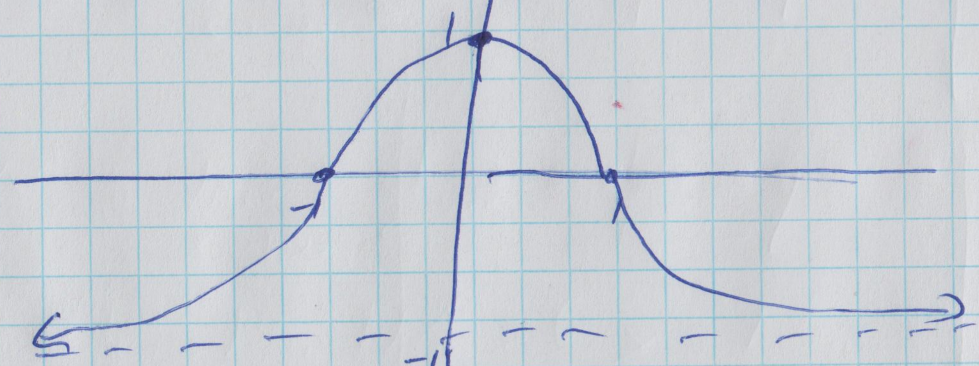
(Could use l'Hôpital's Rule...)

Since $h(x)$ is an even function, ie $h(x) = h(-x)$.

$$\left[h(-x) = \frac{1-(-x)^2}{1+(-x)^2} = \frac{1-x^2}{1+x^2} = h(x)\right],$$

the graph of $h(x)$ is symmetric about the y-axis, so $h(x)$ also has a H.A. of $y=-1$ in the +'ve direction.

<u>Graph:</u>



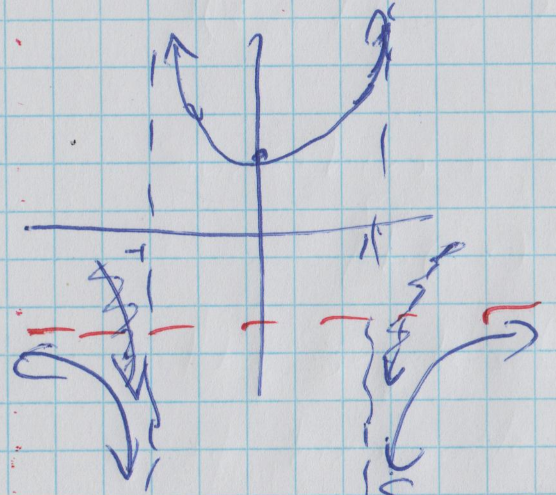
4° $\quad i(x) = \dfrac{1+x^2}{1-x^2}$

Note that this function is even:

$$i(-x) = \frac{1-(-x)^2}{1+(-x)^2} = \frac{1-x^2}{1+x^2} = i(x).$$

$i(x)$ is defined and continuous for all x except for $x = \pm 1$, because these make $1-x^2=0$.

<u>V.A.</u>: $\displaystyle\lim_{x\to -1^-} \frac{1+x^2 \to 2}{1-x^2 \to 0^-} = -\infty$ $\quad$ As $x \to -1^-$ (so $x<-1$, so $|x|>1$), $1-x^2 \to 0^-$ (so $1-x^2 = 1-|x|^2 < 0$)



Since $i(x)$ is even, $\displaystyle\lim_{x\to 1^+} \frac{1+x^2}{1-x^2} = \lim_{x\to -1^-}\frac{1+x^2}{1-x^2} = -\infty.$

$\displaystyle\lim_{x\to 1^-} \frac{1+x^2 \to 2}{1-x^2 \to 0^+} = +\infty$ $\quad$ As $x\to 1^-$ (so $x<1$, so $1-x^2>0$), $1-x^2 \to 0^+$

Since $i(x)$ is even, $\displaystyle\lim_{x\to -1^+} \frac{1+x^2}{1-x^2} = \lim_{x\to 1^-} \frac{1+x^2}{1-x^2} = +\infty$

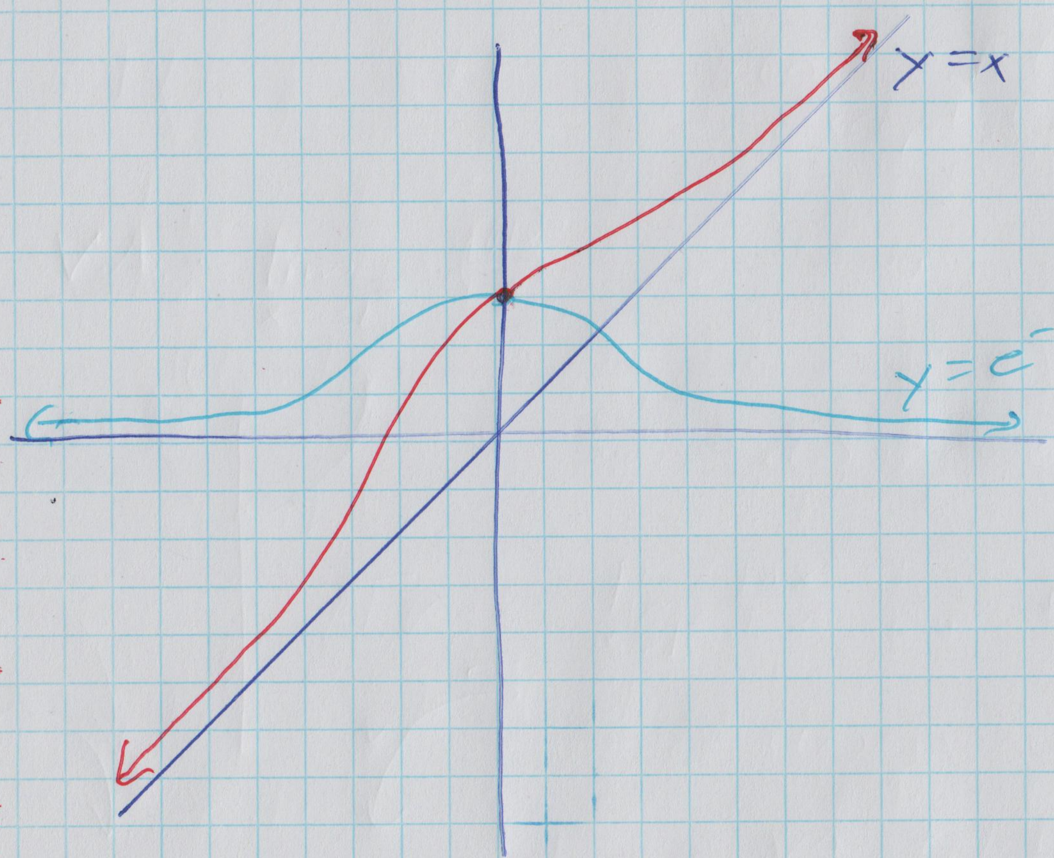
So we have a VA at both $x=1$ and $x=-1$, and the function goes to opposite infinities at each one.

<u>H.A.</u>: $\displaystyle\lim_{x\to\infty} \frac{1+x^2 \to \infty}{1-x^2 \to -\infty} = \lim_{x\to\infty} \frac{\frac{d}{dx}(1+x^2)}{\frac{d}{dx}(1-x^2)} = \lim_{x\to\infty} \frac{\cancel{2x}}{-\cancel{2x}} = -1$

Since $i(x)$ is even, it follows that $\displaystyle\lim_{x\to-\infty} \frac{1+x^2}{1-x^2} = -1$, too.

So we have a HA of $y=-1$ in both directions.

5° $j(x) = x + e^{-x^2}$ Graphically only!



$j(x) = x + e^{-x^2}$

has a "slant asymptote" of $y = x$ as $x \to \pm\infty$

Asymptotic behaviours are not just for horizontal and vertical lines!