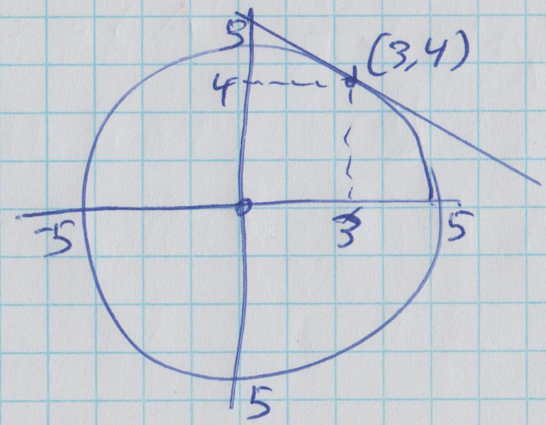
2020-09-30 ①

Derivatives V - Implicit differentiation

(or, the unholy union of calculus and algebra)
(or, when you don't want to solve for y)



eg Consider the circle $x^2+y^2=25$.
What is the slope of the tangent line to this circle at $(3,4)$?

1° Conventional method: a) Solve for y as a function of x near $(3,4)$.
b) Take the derivative & plug in $x=3$.

a) $x^2+y^2=25$

$\Rightarrow y^2 = 25-x^2$

$\Rightarrow y = \pm\sqrt{25-x^2}$

near $(3,4)$, we need the positive root, so

$y=\sqrt{25-x^2}$

b) $\frac{dy}{dx} = \frac{d}{dx}\sqrt{25-x^2} = \frac{d}{dx}(25-x^2)^{1/2}$

$= \frac{1}{2}(25-x^2)^{1/2-1}\cdot\frac{d}{dx}(25-x^2)$

$= \frac{1}{\cancel{2}}(25-x^2)^{-1/2}\cdot(-\cancel{2}x)$

$= \frac{-x}{\sqrt{25-x^2}}$ so $\left.\frac{dy}{dx}\right|_{x=3} = \frac{-3}{\sqrt{25-3^2}}$

$= \frac{-3}{\sqrt{25-9}} = \frac{-3}{\sqrt{16}}$

$= -3/4$.

2° Implicit differentiation: a) differentiate $x^2+y^2=25$

b) solve for $\frac{dy}{dx}$ & the plug in $x=3$ & $y=4$.

a) $\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25) = 0$

$= 2x + \frac{d}{dx}(y^2) = 2x + \frac{dy^2}{dy}\cdot\frac{dy}{dx}$

$= 2x + 2y\cdot\frac{dy}{dx}$

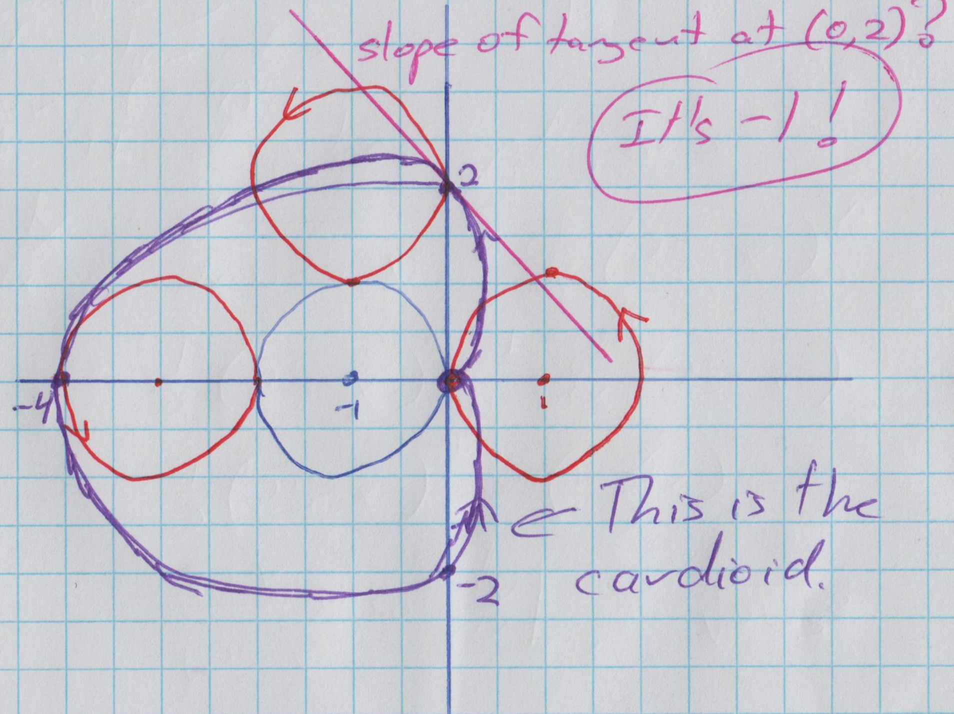
b) $\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}y} = -\frac{x}{y}$

$\left.\frac{dy}{dx}\right|_{x=3\ \&\ y=4} = -\frac{3}{4}$

eg Consider the cardioid

$(x^2+y^2)^2 + 4x(x^2+y^2) - 4y^2 = 0$

We'll try to compute the slope of the tangent to the cardioid at $(x,y) = (0,2)$.



$$\frac{d}{dx}\left[(x^2+y^2)^2+4x(x^2+y^2)-4y^2\right]=\frac{d}{dx}(0)=0$$

$$=2(x^2+y^2)\frac{d}{dx}(x^2+y^2)+4\left[\frac{dx}{dx}\right]\cdot(x^2+y^2)+4x\left[\frac{d}{dx}(x^2+y^2)\right]-4\cdot\frac{d}{dx}y^2$$

$$=2(x^2+y^2)\cdot\left(\frac{d}{dx}x^2+\frac{d}{dx}y^2\right)+4\cdot1\cdot(x^2+y^2)+4x\left(\frac{dx^2}{dx}+\frac{dy^2}{dx}\right)-4\cdot\frac{dy^2}{dy}\cdot\frac{dy}{dx}$$

$$=2(x^2+y^2)\cdot\left(2x+\frac{dy^2}{dy}\cdot\frac{dy}{dx}\right)+4(x^2+y^2)+4x\left(2x+\frac{dy^2}{dy}\cdot\frac{dy}{dx}\right)-4\cdot2y\cdot\frac{dy}{dx}$$

$$=4x(x^2+y^2)+2(x^2+y^2)\cdot2y\cdot\frac{dy}{dx}+4(x^2+y^2)+8x^2+4x\cdot2y\cdot\frac{dy}{dx}-8y\cdot\frac{dy}{dx}$$

$$=\left[4x(x^2+y^2)+8x^2+4(x^2+y^2)\right]+\left[4y(x^2+y^2)+8xy-8y\right]\frac{dy}{dx}$$

$$\therefore\ \frac{dy}{dx}=\frac{-\left[4x(x^2+y^2)+8x^2+4(x^2+y^2)\right]}{4y(x^2+y^2)+8xy-8y}$$

$$\left.\frac{dy}{dx}\right|_{(x,y)=(0,2)}=\frac{-\left[\cancel{4\cdot0\cdot(0^2+2^2)}+\cancel{8\cdot0^2}+4(\cancel{0^2}+2^2)\right]}{4\cdot2\cdot(\cancel{0^2}+2^2)+\cancel{8\cdot0\cdot2}-8\cdot2}$$

$$=\frac{-4\cdot2^2}{4\cdot2\cdot2^2-8\cdot2}=\frac{-16}{32-16}=\frac{-16}{16}=-1$$