Derivatives II - Rules and examples, continued.

Our objective of building up a set of rules for derivatives, and library of particular derivatives that will let us find the derivative of most functions we're likely to see.

Recap: Rules:

0° [Easy constant rule] $\frac{d}{dx} c = 0$

1° Constant Rule [Multiplication by Constant] $\frac{d}{dx}(cy) = c\left(\frac{dy}{dx}\right)$ or $(cf)'(x) = c f'(x)$

2° Sum Rule $\frac{d}{dx}(f(x)+g(x)) = \left(\frac{d}{dx} f(x)\right) + \left(\frac{d}{dx} g(x)\right)$

or $(f+g)'(x) = f'(x) + g'(x)$

3° Power Rule $\frac{d}{dx} x^a = a x^{a-1}$ or $(x^a)' = a x^{a-1}$

[works for all $a \in \mathbb{R}$]

Library: 0° $\frac{d}{dx} c = 0$

1° $\frac{d}{dx} x^a = a x^{a-1}$

2° $\frac{d}{dx} \sin(x) = \cos(x)$ $\quad$ $\sin'(x) = \cos(x)$

3° $\frac{d}{dx} e^x = e^x$

On to more rules:

4° Product Rule: $(fg)'(x) = f'(x)\,g(x) + f(x)\,g'(x)$

or $\frac{d}{dx}\left(f(x)g(x)\right) = \left(\frac{d}{dx} f(x)\right) g(x) + f(x) \left(\frac{d}{dx} g(x)\right)$

proof: $(fg)'(x) = \lim_{h \to 0} \frac{(fg)(x+h) - (fg)(x)}{h}$

$- f(x)g(x)$

$= \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h} = \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x+h) + f(x)\,g(x+h)}{h}$

$= \lim_{h \to 0} \left[\frac{f(x+h)\,g(x+h) - f(x)\,g(x+h)}{h} + \frac{f(x)\,g(x+h) - f(x)\,g(x)}{h} \right]$

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$$= \left[\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h}\right] + \left[\lim_{h\to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h}\right]$$

$$= \left[\lim_{h\to 0} \frac{(f(x+h) - f(x))}{h} g(x+h)\right] + f(x)\left[\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right]$$

$$= \left[\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right]\left[\lim_{h\to 0} g(x+h)\right] + f(x)g'(x)$$

$$= f'(x)g(x) + f(x)g'(x)$$

Note that $\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \to 0$

can only make sense if

$g(x+h) - g(x) \to 0$ as $h \to 0$

ie $\lim_{h\to 0} g(x+h) = g(x)$.

Example:

$$\frac{d}{dx}(xe^x) = \left(\frac{dx}{dx}\right)e^x + x\left(\frac{de^x}{dx}\right)$$

$$= (1x^0)e^x + xe^x$$

$$= e^x + xe^x = (1+x)e^x$$

Consequence: If a function is differentiable at x, then it is continuous at x.

So if $f(x)$ is not continuous at x, it isn't differentiable at x.

5° Quotient Rule

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Note that $g(x) \neq 0$ for this to make sense.

or

$$\frac{d(u/v)}{dx} = \frac{\left(\frac{d}{dx}u\right)\cdot v - u\cdot\left(\frac{dv}{dx}\right)}{v^2}$$

Example: a)

$$\frac{d}{dx}\left(\frac{\sin(x)}{x}\right) = \frac{\left(\frac{d}{dx}\sin(x)\right)x - \sin(x)\left(\frac{d}{dx}x\right)}{x^2}$$

$$= \frac{\cos(x)\cdot x - \sin(x)\cdot 1}{x^2}$$

$$= \frac{x\cos(x) - \sin(x)}{x^2} = \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}$$

b)

$$\frac{d}{dx}x^{-3} = \frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{\left(\frac{d}{dx}1\right)\cdot x^3 - 1\cdot\left(\frac{d}{dx}x^3\right)}{(x^3)^2} = \frac{\cancel{0\cdot x^3} - 3x^2}{x^6} = \frac{-3}{x^4}$$

Power Rule for positive n

$$= -3x^{-4}$$

(general) Power Rule

$$\frac{d}{dx}x^{-3} = -3x^{-3-1} = -3x^{-4}$$

Example: $\frac{d}{dx}\left(\frac{x+1}{x^2+1}\left(e^x+\sin(x)\right)\right)$

Product Rule

$$= \left[\frac{d}{dx}\left(\frac{x+1}{x^2+1}\right)\right]\cdot\left(e^x+\sin(x)\right) + \left(\frac{x+1}{x^2+1}\right)\cdot\left[\frac{d}{dx}\left(e^x+\sin(x)\right)\right]$$

Quotient Rule | Sum Rule

$$= \left[\frac{\left[\frac{d}{dx}(x+1)\right](x^2+1) + (x+1)\left[\frac{d}{dx}(x^2+1)\right]}{(x^2+1)^2}\right]\left(e^x+\sin(x)\right) + \left(\frac{x+1}{x^2+1}\right)\left[\left(\frac{d}{dx}e^x\right)+\left(\frac{d}{dx}\sin(x)\right)\right]$$

Sum Rule | Library!

$$= \left[\frac{\left[\left(\frac{d}{dx}x\right)+\left(\frac{d}{dx}1\right)\right](x^2+1) + (x+1)\left[\left(\frac{dx^2}{dx}\right)+\left(\frac{d}{dx}1\right)\right]}{(x^2+1)^2}\right]\left(e^x+\sin(x)\right) + \left(\frac{x+1}{x^2+1}\right)\left[e^x+\cos(x)\right]$$

Power Rule (& easy constant rule)

$$= \frac{[1+0](x^2+1) + (x+1)[2x+0]}{(x^2+1)^2}\left(e^x+\sin(x)\right) + \left(\frac{x+1}{x^2+1}\right)\left[e^x+\cos(x)\right]$$

$$= \frac{x^2+1+2x^2+2x}{(x^2+1)^2}\left(e^x+\sin(x)\right) + \left(\frac{x+1}{x^2+1}\right)\left[e^x+\cos(x)\right]$$

$$= \frac{3x^2+2x+1}{(x^2+1)^2}\left(e^x+\sin(x)\right) + \left(\frac{x+1}{x^2+1}\right)\left(e^x+\cos(x)\right)$$

Next Chain Rule!