Limits III - Practical rules for computing limits

The ε-δ definition allows one to verify whether a limit is correct or not, but doesn't give a clue how to find the limit in the first place.

$0°$ $\lim\limits_{x\to a} x = a$ [Use $\delta = \varepsilon$...]

$1°$ (Sum Rule for limits) $\lim\limits_{x\to a}(f(x)+g(x)) = \left(\lim\limits_{x\to a} f(x)\right) + \left(\lim\limits_{x\to a} g(x)\right)$

[provided that all three limits exist!]

<u>proof:</u> Suppose that all three limits exist [ie satisfy ε-δ def'n] and that $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$.

We need to verify that $\lim\limits_{x\to a}(f(x)+g(x)) = L+M$,

<u>ie</u> For all $\varepsilon>0$, there is a $\delta>0$, such that (for all x) if $|x-a|<\delta$, then $|(f(x)+g(x)) - (L+M)| < \varepsilon$,

Use reverse-engineering to find the $\delta > 0$ for a given $\varepsilon > 0$:

$$|(f(x)+g(x)) - (L+M)| < \varepsilon$$

$\left[\ |a+b| \leq |a|+|b|\ \right]$

$$\Longleftrightarrow |(f(x)-L)+(g(x)-M)| < \varepsilon$$

Observe that $|(f(x)-L)+(g(x)-M)| \leq |f(x)-L| + |g(x)-M|$

$\left[< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \right]$

so it would be good enough to get

$$|f(x)-L| < \frac{\varepsilon}{2} \quad \text{and} \quad |g(x)-M| < \frac{\varepsilon}{2}$$

Since $\lim_{x \to a} f(x) = L$ and $\frac{\varepsilon}{2} > 0$, there is some $\delta_1 > 0$, such that if $|x-a| < \delta_1$, then $|f(x)-L| < \frac{\varepsilon}{2}$.

Since $\lim_{x \to a} g(x) = M$ and $\frac{\varepsilon}{2} > 0$, there is some $\delta_2 > 0$, such that if $|x-a| < \delta_2$, then $|g(x)-M| < \frac{\varepsilon}{2}$.

Let $\delta = \min(\delta_1, \delta_2)$. If we do so:

then, if $|x-a|<\delta$, we get

$$|x-a|<\delta\le\delta_1, \text{ so } |f(x)-L|<\frac{\varepsilon}{2}$$

$$\& \quad |x-a|<\delta\le\delta_2, \text{ so } |g(x)-M|<\frac{\varepsilon}{2}.$$

Then $|(f(x)+g(x))-(L+M)|$

$$= |(f(x)-L)+(g(x)-M)|$$

$$\le |f(x)-L|+|g(x)-M|$$

$$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon, \text{ as desired.}$$

$$\therefore \lim_{x\to a}(f(x)+g(x)) = L+M = \left(\lim_{x\to a} f(x)\right)+\left(\lim_{x\to a} g(x)\right).$$

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end of proof symbol (others: $\Box$, $\blacksquare$, Q.E.D.)

eg $$\lim_{x\to 1}(2x) = \lim_{x\to 1}(x+x) \overset{\text{rule 1}}{=} \left(\lim_{x\to 1} x\right)+\left(\lim_{x\to 1} x\right) \overset{\text{rule 0}}{=} 1+1 = 2$$

2° (Constant rule for limits) If C is a constant,

$$\lim_{x \to a} C = C. \qquad [\text{Use any } \delta > 0 \text{ you like...}]$$

3° (Product Rule for limits)

$$\lim_{x \to a} [f(x) g(x)] = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$$

[provided all the limits in question exist]

eg $$\lim_{x \to -2} x^2 = \lim_{x \to -2} x \cdot x \overset{\text{rule 3}}{=} \left(\lim_{x \to -2} x\right)\left(\lim_{x \to -2} x\right) \overset{\text{rule 0}}{=} (-2)(-2) = 4$$

4° (Take a constant out rule) C a constant

$$\lim_{x \to a} C f(x) = C\left[\lim_{x \to a} f(x)\right] \qquad [\text{provided both limits exist}]$$

$\downarrow$ "

$$\left(\lim_{x \to a} C\right)\left(\lim_{x \to a} f(x)\right) \nearrow "$$

5° (Quotient Rule for limits)

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

[Need all three limits to exist, and $\lim_{x\to a} g(x) \neq 0$]

Cheap example for how this could go wrong if $\lim_{x\to a} g(x) = 0$:

$$1 = \lim_{x\to 0} 1 = \lim_{x\to 0} \frac{x}{x} \overset{?}{=} \frac{\lim_{x\to 0} x}{\lim_{x\to 0} x} = \frac{0}{0} \text{ undefined}$$

6. (Squeeze Theorem) If $f(x) \le h(x) \le g(x)$ for all x near a, and

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} g(x),$$

then $\lim_{x\to a} h(x) = L$ too.

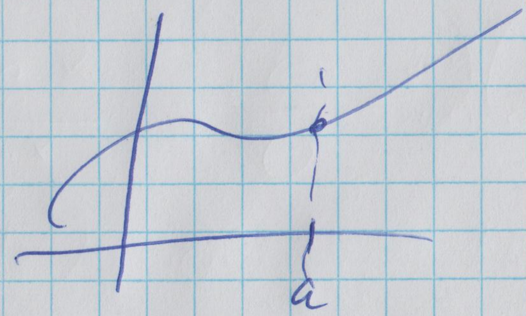
as $x \to a$, $f(x) \le h(x) \le g(x)$

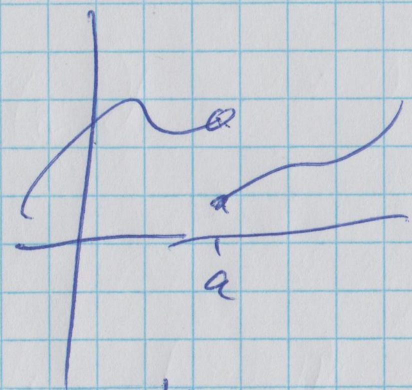
$$L$$

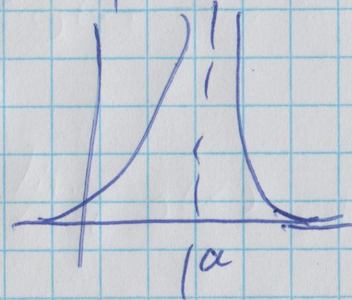
<u>Def'n:</u> $f(x)$ is continuous at a if $\lim_{x \to a} f(x) = f(a)$.

[Note: the limit needs to exist & $f(a)$ has to be defined.]

Informally, $f(x)$ is continuous at a if you can draw the graph at a without lifting your pen off the paper







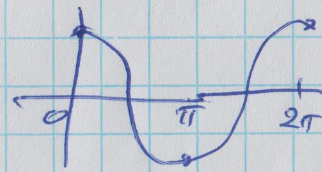
<u>Def'n:</u> $f(x)$ is continuous (on some interval) if it is continuous at every point (of that interval)

7° (Continuity Rule for limits)

If $f(x)$ is continuous, then $\lim\limits_{x \to a} f(x) = f(a)$.

Most functions are continuous most of the time.

eg $\lim\limits_{x \to 3\pi} e^{\cos(x)} = e^{\cos(3\pi)} = e^{-1} = \frac{1}{e}$



8° (Composition Rule for limits)

Suppose $\lim\limits_{x \to a} g(x) = L$ and $\lim\limits_{t \to L} f(t) = M$.

Then $\lim\limits_{x \to a} f(g(x)) = \lim\limits_{x \to a} (f \circ g)(x) = M$.

<u>Corollary</u>: A composition of continuous functions is continuous.

Next time: examples of computing limits!