Limits II - non-limits and limits to infinity

eg An example of showing a limit isn't right using the ε-δ defn.

$$\lim_{x \to 2} (-2x+5) \neq 3$$

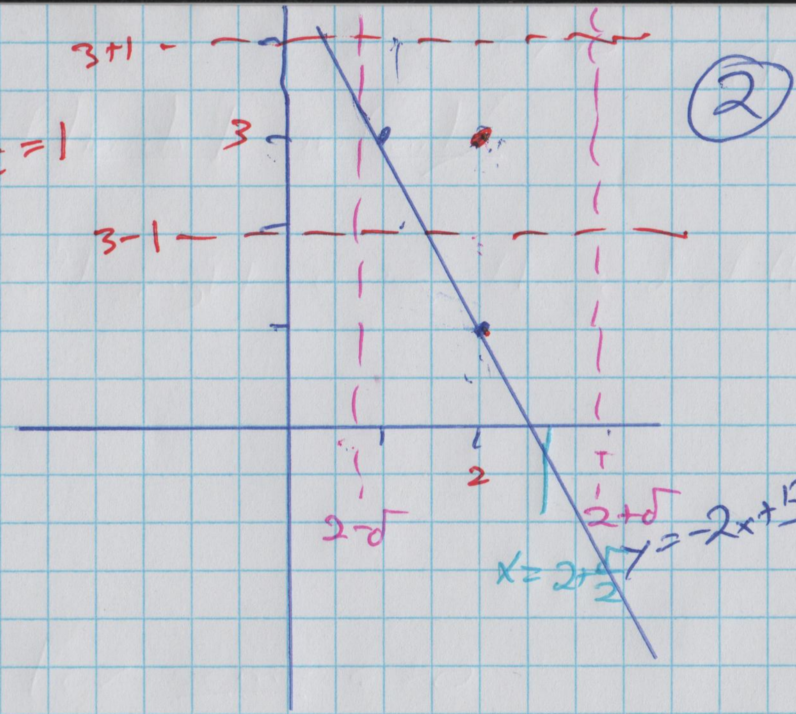
We have to show:

Not [For ~~every~~ $\varepsilon > 0$
There is a $\delta > 0$
such that
~~if~~ for all x
if $|x-2| < \delta$
then $|(-2x+5)-3| < \varepsilon$.]

$\Rightarrow$ For some $\varepsilon > 0$
Not [There is a $\delta > 0$
such that for all x
if $|x-2| < \delta$
then $|(-2x+5)-3| < \varepsilon$]

$\Rightarrow$ For some $\varepsilon > 0$
there is no $\delta > 0$
such that Not [
for all x, if $|x-2| < \delta$,
then $|(-2x+5)-3| < \varepsilon$]

$\Rightarrow$ For some $\varepsilon > 0$, ~~there is no~~ for all $\delta > 0$,
there is some x such that
Not [if $|x-2| < \delta$,
then $|(-2x+5)-3| < \varepsilon$]



$\Rightarrow$ For some $\varepsilon > 0$, for all $\delta > 0$, there is some x such that $|x-2| < \delta$ but $|(-2x+5)-3| \geq \varepsilon$.

Try $\varepsilon = 1$. Suppose we are given a $\delta > 0$. We need to find an x such that $|x-2| < \delta$ but $|(-2x+5)-3| \geq 1$. $=\varepsilon$

Then try $x = 2 + \frac{\delta}{2}$. Then $|2 + \frac{\delta}{2} - 2| = |\frac{\delta}{2}| < \delta$,

and $|(-2\cdot(2+\frac{\delta}{2}) + 5) - 3| = |-4 - \delta + 5 - 3| = |-2-\delta| = 2+\delta \geq 1$

since $\delta > 0$.

$\therefore$ According to the ε-δ definition of limits

$\lim_{x \to 2} (-2x+5) \neq 3$. [The better way to do this would be to show that $\lim_{x \to 2} (-2x+5) = 1 \neq 3$.]

Limits as x goes to infinity.

(∞ is the shorthand symbol for "infinity". ∞ is <u>not</u> a real number.)

<u>Defn:</u> $\lim_{x\to\infty} f(x) = L$

means that "For every $\varepsilon > 0$, there is an $N > 0$, such that $|f(x) - L| < \varepsilon$ for all $x \geq N$."

[& $\lim_{x\to-\infty} f(x) = L$ means $\lim_{x\to\infty} f(-x) = L$.]

<u>eg</u> verify $\lim_{x\to\infty} \frac{x+1}{x} = 1$ i.e. Check that for any given $\varepsilon > 0$, we can find an $N > 0$ such that if $x \geq N$, then $\left|\frac{x+1}{x} - 1\right| < \varepsilon$.

As usual we try to reverse-engineer the N from the ε.

$$\left|\frac{x+1}{x} - 1\right| < \varepsilon \iff \left|\frac{x}{x} + \frac{1}{x} - 1\right| < \varepsilon \iff \left|\cancel{1} + \frac{1}{x} \cancel{-1}\right| < \varepsilon$$

$$\iff \left|\frac{1}{x}\right| < \varepsilon \iff \frac{1}{x} < \varepsilon \iff 1 < \varepsilon x \iff \frac{1}{\varepsilon} < x$$

$$\iff x \geq N > \frac{1}{\varepsilon}$$

Thus $N > \frac{1}{\varepsilon}$ will do the job (since every step is reversible).

Verify

e.g. $\lim_{x \to -\infty} \dfrac{\sin(x)}{x} = 0$ i.e. Check that for any $\varepsilon > 0$ we can find an $N > 0$ such that if $x \geq N$, then $\left| \dfrac{\sin(x)}{-x} - 0 \right| < \varepsilon$.

i.e. $\lim_{x \to \infty} \dfrac{\sin(-x)}{-x} = 0$

Reverse-engineering again: $\left| \dfrac{\sin(x)}{-x} - 0 \right| < \varepsilon$

$\Longleftrightarrow \left| \dfrac{\sin(x)}{-x} \right| < \varepsilon \Longleftrightarrow \left| \dfrac{-\sin(x)}{-x} \right| < \varepsilon \Longleftrightarrow \left| \dfrac{\sin(x)}{x} \right| < \varepsilon$

$\Longleftrightarrow \dfrac{|\sin(x)|}{x} < \varepsilon \Longleftrightarrow |\sin(x)| < \varepsilon x$ but $|\sin(x)| \leq 1$

So if we ensure that $\varepsilon x > 1$, then $\varepsilon x > 1 \geq |\sin(x)|$

which is enough to get the fully reversible steps.

So pick $N \geq \dfrac{1}{\varepsilon}$.

Then if $x \geq N$, then $x > \dfrac{1}{\varepsilon} \Longleftrightarrow \varepsilon x > 1$ ... $\left| \dfrac{\sin(-x)}{-x} - 0 \right| < \varepsilon$.

$\therefore \lim_{x \to -\infty} \dfrac{\sin(x)}{x} = 0$.

Algebraic rules for manipulating limits (without directly using ε-δ or ε-N) next time.