Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Fall 2019

Solutions to Assignment #4
Not the Zero Function

The following function was used as an example by Augustin-Louis Cauchy when investigating the convergence of Taylor series.

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

1. Verify that $f(x)$ is continuous at $x = 0$. *[4]*

SOLUTION. We need to check that $\lim_{x\to 0} f(x) = f(0)$. Observe that that as $x \to 0$, we have $x^2 \to 0^+$, so $1/x^2 \to +\infty$, and hence $-1/x^2 \to -\infty$. It follows that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{-1/x^2} = \lim_{t\to -\infty} e^t = 0 = f(0)\,,$$

as desired. □

2. Show that $f'(0)$ is defined and equal to 0. *[6]*

SOLUTION. Note that $f(x)$ is defined differently at $x = 0$ than it is for all other points, which makes it difficult to rely on either definition of $f(x)$ to compute $f'(0)$ in the usual way. We will avoid that problem by going back to the limit definition of the derivative, $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$, to compute $f'(0)$.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{e^{-1/(0+h)^2} - 0}{h} = \lim_{h\to 0} \frac{e^{-1/h^2}}{h} = \lim_{h\to 0} \frac{1}{h} \cdot e^{-(1/h)^2}$$

How do we proceed from here?

Even though we have $\dfrac{e^{-1/h^2}}{h} \begin{matrix} \to 0 \\ \to 0 \end{matrix}$ as $h \to 0$, it is not a good idea to use l'Hôpital's Rule here. Sadly, $\dfrac{\frac{d}{dh} e^{-1/h^2}}{\frac{d}{dh} h}$ works out to $\dfrac{2e^{-1/h^2}}{h^3}$, which is worse than what we started with.

A more promising idea is to use the substitution $t = \dfrac{1}{h}$ to convert $\dfrac{1}{h} \cdot e^{-(1/h)^2}$ into the easier-to-handle $te^{-t^2} = \dfrac{t}{e^{t^2}}$. This does have one complication, though: as $h \to 0$, $t = \dfrac{1}{h} \to +\infty$ or $-\infty$ depending on whether $h \to 0^+$ or 0^-, respectively. This means we have to compute two limits and hope they work out the same way. First, we compute the limit as $h \to 0^+$:

$$\lim_{h\to 0^+} \frac{1}{h} \cdot e^{-(1/h)^2} = \lim_{t\to +\infty} \frac{t}{e^{t^2}} \begin{matrix} \to +\infty \\ \to +\infty \end{matrix} \quad \text{so, by l'Hôpital's Rule,}$$

$$= \lim_{t\to +\infty} \frac{\frac{d}{dt} t}{\frac{d}{dt} e^{t^2}} = \lim_{t\to +\infty} \frac{1}{2te^{t^2}} \begin{matrix} \to 1 \\ \to +\infty \end{matrix} = 0$$

Second, we compute the limit as $h \to 0^-$:

$$\lim_{h\to 0^-} \frac{1}{h} \cdot e^{-(1/h)^2} = \lim_{t\to-\infty} \frac{t \;\; \to -\infty}{e^{t^2} \to +\infty} \quad \text{so, by l6Hôpital's Rule,}$$

$$= \lim_{t\to-\infty} \frac{\frac{d}{dt}t}{\frac{d}{dt}e^{t^2}} = \lim_{t\to-\infty} \frac{1 \quad \to 1}{2te^{t^2} \to -\infty} = 0$$

Since $\lim_{h\to 0^+} \frac{1}{h} \cdot e^{-(1/h)^2} = 0 = \lim_{h\to 0^-} \frac{1}{h} \cdot e^{-(1/h)^2}$, we have that $\lim_{h\to 0} \frac{1}{h} \cdot e^{-(1/h)^2}$ exists and $= 0$.

Thus $f'(0) = \lim_{h\to 0} \frac{1}{h} \cdot e^{-(1/h)^2} = 0$, as desired. ■

Note. It turns out that the second, third, fourth – every! – derivative of $f(x)$ is defined and equal to 0 at $x = 0$, making it indistinguishable from the zero function, $g(x) = 0$ for all x, as far as far as calculus can determine it from its behavious at $x = 0$.