Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals (Section A)
TRENT UNIVERSITY, Fall 2019
Solutions to the Quizzes

Quiz #1. *Wednesday, 18 September. [7 minutes]*

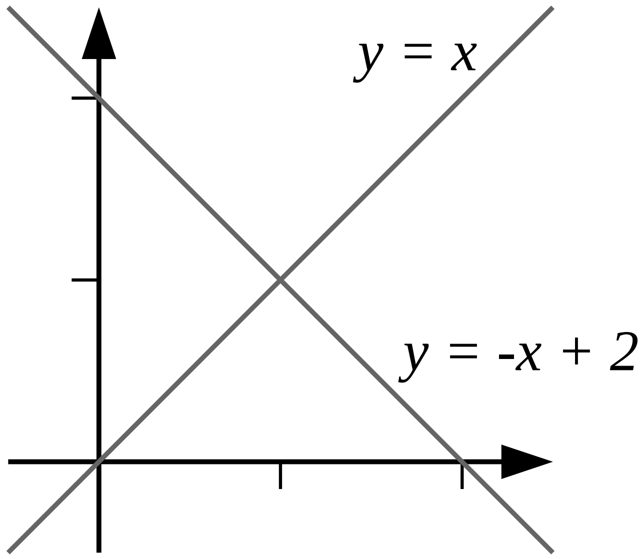
Consider the line $y = -x + 2$.

1. Find the equation of the line through $(2, 2)$ that is perpendicular to the given line. *[3]*
2. Sketch the graphs of both of these lines. *[2]*

SOLUTION. 1. A line $y = mx + b$ that is perpendicular to the given line must have a slope that is the negative reciprocal of the slope of the given line, so $m = -\frac{1}{-1} = 1$. Thus the perpendicular line has equation $y = x + b$, where it remains to determine b.

Since the point $(2, 2)$ is on the perpendicular line, we must have $2 = 2 + b$, so $b = 2 - 2 = 0$. Thus the line perpendicular to the given line that passes through $(2, 2)$ is given by the equation $y = x$. □

2. Here is a sketch of these lines:



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Quiz #2. *Wednesday, 25 September. [10 minutes]*

Compute both of the following limits.

1. $\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1}$ *[2.5]* 2. $\lim_{x \to 0} \frac{\sin(2x)}{x}$ *[2.5]*

SOLUTION. 1. $\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+2)}{x - 1} = \lim_{x \to 1} (x + 2) = 1 + 2 = 3$ □

2. A bit of algebra, the fact that $2x \to 0$ as $x \to 0$, and substituting t for $2x$ near the end:

$$\lim_{x \to 0} \frac{\sin(2x)}{x} = \lim_{x \to 0} \frac{2}{2} \cdot \frac{\sin(2x)}{x} = \lim_{x \to 0} \frac{2\sin(2x)}{2x} = 2 \lim_{x \to 0} \frac{\sin(2x)}{2x}$$
$$= 2 \lim_{2x \to 0} \frac{\sin(2x)}{2x} = 2 \lim_{t \to 0} \frac{\sin(t)}{t} = 2 \cdot 1 = 2 \quad ■$$

Quiz #3. *Wednesday, 2 October. [10 minutes]*

Compute the derivatives of both of the following functions, simplifying where you can.

1. $f(x) = \dfrac{x^2-2}{x-1}$ *[2.5]* 2. $g(x) = \sqrt{1+\tan^2(x)}$ *[2.5]*

SOLUTION. 1. *[Quotient Rule & algebra]*

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{x^2-2}{x-1}\right) = \frac{\left[\frac{d}{dx}\left(x^2-2\right)\right]\cdot(x-1) - \left(x^2-2\right)\cdot\left[\frac{d}{dx}(x-1)\right]}{(x-1)^2} \\ &= \frac{[2x]\cdot(x-1)-\left(x^2-2\right)\cdot[1]}{(x-1)^2} = \frac{2x^2-2x-x^2+2}{(x-1)^2} = \frac{x^2-2x+2}{(x-1)^2} \\ &= \frac{(x-1)^2+1}{(x-1)^2} = 1 + \frac{1}{(x-1)^2} \end{aligned}$$

The last line is just showing off ... :-) □

2. *[Chain Rule & simplify, or, I forgot about $1+\tan^2(x) = \sec^2(x)$]*

$$\begin{aligned} g'(x) &= \frac{d}{dx}\sqrt{1+\tan^2(x)} = \frac{d}{dx}\left(1+\tan^2(x)\right)^{1/2} = \frac{1}{2}\left(1+\tan^2(x)\right)^{-1/2}\cdot\frac{d}{dx}\left(1+\tan^2(x)\right) \\ &= \frac{1}{2}\left(1+\tan^2(x)\right)^{-1/2}\cdot 2\tan(x)\cdot\frac{d}{dx}\tan(x) = \left(1+\tan^2(x)\right)^{-1/2}\tan(x)\sec^2(x) \\ &= \frac{\tan(x)\sec^2(x)}{\sqrt{1+\tan^2(x)}} \qquad \square \end{aligned}$$

2. *[Simplify first, or, I did remember that $1+\tan^2(x) = \sec^2(x)$]*

$$g'(x) = \frac{d}{dx}\sqrt{1+\tan^2(x)} = \frac{d}{dx}\sqrt{\sec^2(x)} = \frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

This is the same answer as above: just plug in the identity $1+\tan^2(x) = \sec^2(x)$ in the final answer above and simplify away. Of course, it could be rewritten in many different ways, given the multitude of trig idenitities out there. ■

Quiz #4. *Wednesday, 9 October. [10 minutes]*

Compute the derivatives of both of the following functions, simplifying where you can.

1. $f(x) = \log_2(3^x)$ *[2.5]* 2. $g(x) = \ln(\sec(x) + \tan(x))$ *[2.5]*

SOLUTION. 1. *(Simplify first.)* $f(x) = \log_2(3^x) = x\log_2(3)$, so $f'(x) = \log_2(3)$. □

1. *(Differentiate first.)* Chain Rule all the way:

$$f'(x) = \frac{d}{dx}\log_2(3^x) = \frac{1}{\ln(2)\cdot 3^x}\cdot\left[\frac{d}{dx}3^x\right] = \frac{1}{\ln(2)\cdot 3^x}\cdot\ln(3)\cdot 3^x = \frac{\ln(3)}{\ln(2)} = \log_2(3) \quad \square$$

2. There is nothing for it but to differentiate away using the Chain Rule and hope:

$$\begin{aligned}
g'(x) &= \frac{d}{dx}\ln(\sec(x) + \tan(x)) \\
&= \frac{1}{\sec(x) + \tan(x)}\cdot\left[\frac{d}{dx}(\sec(x) + \tan(x))\right] \\
&= \frac{1}{\sec(x) + \tan(x)}\cdot\left[\sec(x)\tan(x) + \sec^2(x)\right] \\
&= \frac{\sec(x)\left[\tan(x) + \sec(x)\right]}{\sec(x) + \tan(x)} = \sec(x) \qquad \blacksquare
\end{aligned}$$

Quiz #5. *Wednesday, 16 October. [20 minutes]*

1. Find the domain and any and all intercepts, vertical and horizontal asymptotes, intervals of increase and decrease, maximum and minimum points, intervals of concavity, and inflection points of $f(x) = xe^x$, and sketch the graph of this function. *[5]*

SOLUTION. We run throught the given checklist:

i. Domain: $f(x) = xe^x$ makes sense for all x, so the domain of $f(x)$ is $\mathbb{R} = (-\infty, \infty)$.

ii. Intercepts. Setting $x = 0$ gives $f(0) = 0e^0 = 0$, so the y-intercept is $y = 0$. Setting $f(x) = xe^x = 0$ tells us that $x = 0$ because $e^x > 0$ for all x, so the x-intercept is $x = 0$.

iii. Vertical asymptotes. Since $f(x)$ is defined and continuous for all x, being the product of the everywhere defined and continuous functions $g(x) = x$ and $h(x) = e^x$, it cannot have any vertical asymptotes.

iv. Horizontal asymptotes. We compute the necessary limits. First, $\lim_{x\to+\infty} xe^x = +\infty$, since $x \to +\infty$ and $e^x \to +\infty$ as $x \to +\infty$. Second, we need to work a bit harder to compute $\lim_{x\to-\infty} xe^x$, since $x \to -\infty$ and $e^x \to 0$ as $x \to -\infty$, and so we need to figure out just what happens in the tug-of-war betwen the two in their product. We rewrite the limit to be able to apply l'Hôpital's Rule:

$$\begin{aligned}
\lim_{x\to-\infty} xe^x &= \lim_{x\to-\infty} \frac{x \ \to -\infty}{e^{-x} \to +\infty} = \lim_{x\to-\infty}\frac{\frac{d}{dx}x}{\frac{d}{dx}e^{-x}} = \lim_{x\to-\infty}\frac{1}{e^{-x}\frac{d}{dx}(-x)} \\
&= \lim_{x\to-\infty}\frac{1 \qquad \to 1}{-e^{-x} \to -(+\infty)} = 0^-
\end{aligned}$$

We thus have no horizontal assymptote on the positive side, and a horizontal asymptote of $y = 0$ on the negative side, which the function approaches from below as $x \to -\infty$.

v. Intervals of increase and decrease, and maximum and minimum points. First, we compute the derivative:

$$f'(x) = \frac{d}{dx}xe^x = \left[\frac{d}{dx}x\right]e^x + x\left[\frac{d}{dx}e^x\right] = 1e^x + xe^x = (1+x)e^x$$

Since $e^x > 0$ for all x, $f'(x)$ is -0, < 0, or > 0 exactly when $1 + x$ is -0, < 0, or > 0, respectively, *i.e.* exactly when x is -1, < -1, or > -1, respectively. It follows that$f(x) = xe^x$ decreases on $(-\infty, -1)$, has a local minimum at $x = -1$, and increases on $(-1, +\infty)$. We summarize these facts in a table:

x	$(-\infty, 0)$	-1	$(-1, +\infty)$
$f'(x)$	$-$	0	$+$
$f(x)$	$\downarrow$	min	$\uparrow$

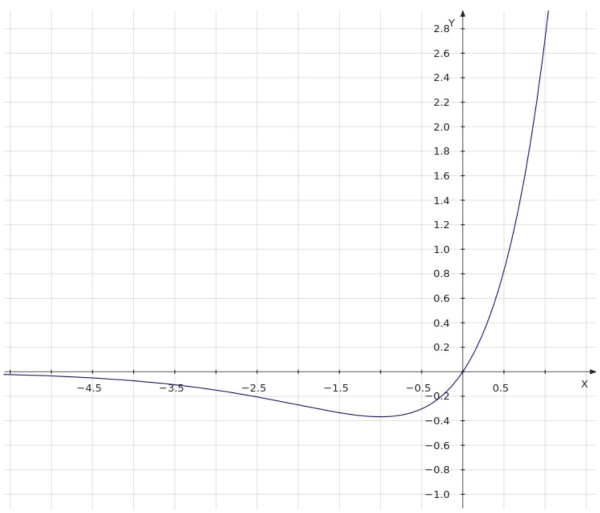
vi. Intervals of concavity and inflection points. First, we compute the second derivative:

$$f''(x) = \frac{d}{dx}(1+x)e^x = \left[\frac{d}{dx}(1+x)\right]e^x + (1+x)\left[\frac{d}{dx}e^x\right] = 1e^x + (1+x)e^x = (2+x)e^x$$

Since $e^x > 0$ for all x, $f''(x)$ is -0, < 0, or > 0 exactly when $2 + x$ is -0, < 0, or > 0, respectively, *i.e.* exactly when x is -2, < -2, or > -2, respectively. It follows that$f(x) = xe^x$ is concave down on $(-\infty, -2)$, has an inflection point at $x = -2$, and is concave up on $(-2, +\infty)$. We summarize these facts in a table:

x	$(-\infty, -2)$	-2	$(-2, +\infty)$
$f''(x)$	$-$	0	$+$
$f(x)$	$\frown$	infl.	$\smile$

vii. The graph. Cheating slightly, we use a graphing program called `kmplot` to draw the graph for us:



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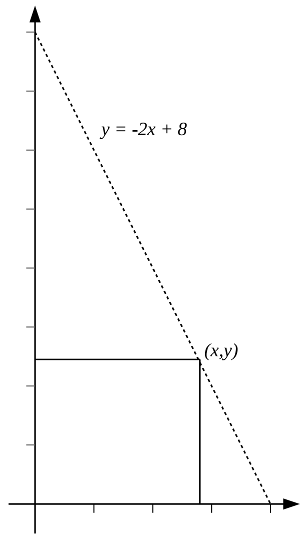
Quiz #6. *Wednesday, 6 November. [10 minutes]*

1. A rectangle with sides parallel to the coordinate axes has one corner at the origin and the opposite corner on the line $y = -2x + 8$ in the first quadrant. Find the maximum area of such a rectangle. *[5]*



SOLUTION. Here's a sketch of the setup:

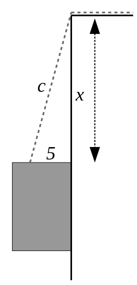
A rectangle with one corner at the origin and the opposite corner at the point (x, y) in the first quadrant, and with its sides parallel to the coordinate axes, has a width of x and a height of y, and hence area $A = xy$. If the point (x, y) is on the line $y = -2x + 8$, we have area $A = xy = x(-2x + 8) = -2x^2 + 8x$ in terms of x. Note for (x, y) to be in the first quadrant and on the line, we must have $0 \le x \le 4$.

To find any critical point(s), observe that

$$\frac{dA}{dx} = \frac{d}{dx}\left(-2x^2 + 8x\right) = -4x + 8,$$

which equals zero exactly when $4x = 8$, *i.e.* when $x = 2$. This is in the interval $0 \le x \le 4$, so we will consider it along with the endpoints: $A(0) = -2 \cdot 0^2 + 8 \cdot 0 = 0$, $A(2) = -2 \cdot 2^2 + 8 \cdot 2 = -8 + 16 = 8$, and $A(4) = -2 \cdot 4^2 + 8 \cdot 4 = -32 + 32 = 0$. The largest of these is clearly $A(2) = 8$, so this is the maximum area of a rectangle with the given constraints. ■

Quiz #7. *Wednesday, 13 November. [15 minutes]*



1. A rectangular block is hauled up a vertical wall by a cable attached to one end of the block so that the end of the cable is always exactly 5 *cubits* from the wall. The other end of the cable goes over the edge of the wall and is being hauled in at a constant rate of $\frac{12}{13}$ *cubits/sec*. At what rate is the block rising at the instant that there are exactly 13 *cubits* of cable between the edge of the wall and the block? *[5]*

SOLUTION. Let c be the length of the cable from the top edge of the wall to where it is attached to the block, and let x be the distance from the top edge of the wall to the block, as in the diagram above. We are told that $\frac{dc}{dt} = -\frac{12}{13}$ and we wish to to know $\frac{dx}{dt}$ at the instant that $x = 12$.

By the Pythagorean Theorem, $c^2 = 5^2 + x^2$, so $x = \sqrt{c^2 - 5^2} = \sqrt{c^2 - 25}$. It follows that

$$\frac{dx}{dt} = \frac{1}{2\sqrt{c^2 - 25}} \cdot \frac{d}{dt}\left(c^2 - 25\right) = \frac{2c}{2\sqrt{c^2 - 25}} \cdot \frac{dc}{dt} = \frac{c}{\sqrt{c^2 - 25}} \cdot \frac{dc}{dt}.$$

When $x = 12$, $c = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13$, so

$$\left.\frac{dx}{dt}\right|_{x=12} = \frac{13}{\sqrt{13^2 - 25}} \cdot \left(-\frac{12}{13}\right) = \frac{13}{\sqrt{169 - 25}} \cdot \left(-\frac{12}{13}\right) = \frac{-12}{\sqrt{144}} = -\frac{12}{12} = -1.$$

Note that the negative sign mean that the distance between the block and the edge of the wall is decreasing, *i.e.* the block is rising at a rate of 1 *cubit/sec*. ■

Quiz #8. *Wednesday, 20 November. [15 minutes]*

Compute each of the following definite integrals.

1. $\displaystyle\int_1^2 \left(x^2 + \frac{1}{x^2}\right)\,dx$ *[2.5]* 2. $\displaystyle\int_0^{\sqrt{\pi/4}} 4x\sec^2\left(x^2\right)\,dx$ *[2.5]*

SOLUTIONS. 1. Basic properties and the Power Rule for integrals, plus arithmetic:

$$\begin{aligned}\int_1^2 \left(x^2 + \frac{1}{x^2}\right)dx &= \int_1^2 \left(x^2 + x^{-2}\right)dx = \left(\frac{x^{2+1}}{2+1} + \frac{x^{-2+1}}{-2+1}\right)\Bigg|_1^2 = \left(\frac{x^3}{3} - x^{-1}\right)\Bigg|_1^2 \\ &= \left(\frac{2^3}{3} - 2^{-1}\right) - \left(\frac{1^3}{3} - 1^{-1}\right) = \left(\frac{8}{3} - \frac{1}{2}\right) - \left(\frac{1}{3} - 1\right) \\ &= \frac{16}{6} - \frac{3}{6} - \frac{2}{6} + \frac{6}{6} = \frac{16-3-2+6}{6} = \frac{17}{6} \qquad \square\end{aligned}$$

2. We will use the substitution $u = x^2$, so $du = 2x\,dx$ and $\begin{matrix} x & 0 & \sqrt{\pi/4} \\ u & 0 & \pi/4 \end{matrix}$.

$$\begin{aligned}\int_0^{\sqrt{\pi/4}} 4x\sec^2\left(x^2\right)dx &= \int_0^{\sqrt{\pi/4}} 2\sec^2\left(x^2\right)\cdot 2x\,dx = \int_0^{\pi/4} 2\sec^2(u)\,du \\ &= 2\tan(u)\big|_0^{\pi/4} \quad \text{because } \frac{d}{du}\tan(u) = \sec^2(u) \\ &= 2\tan\left(\frac{\pi}{4}\right) - 2\tan(0) = 2\cdot 1 - 2\cdot 0 = 2 - 0 = 2 \qquad \blacksquare\end{aligned}$$