Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Fall 2018

Solutions to Assignment #4
It's a cinch!?

Recall from class or the textbook that the basic hyperbolic functions are

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

We can define the other hyperbolic functions from these in the same way that we define the other trigonometric functions from $\sin(x)$ and $\cos(x)$. In particular,

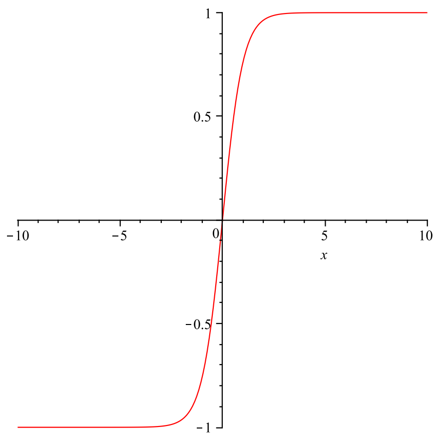
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \text{and} \quad \operatorname{sech}(x) = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}.$$

Like the trigonometric functions, the hyperbolic functions can be inverted, albeit sometimes only partially. The main task in this assignment is to invert $\tanh(x)$.

1. Plot $y = \tanh(x)$. *[1]*

SOLUTION. It helps that `Maple` has $\tanh(x)$ built-in:

```
> plot(tanh(x), x = -10..10)
```



2. What are the domain and range of $\tanh(x)$? *[1]*

SOLUTION. The plot in the solution to question **1** suggests that $\tanh(x)$ is defined for all x, but it's range is confined between $y = -1$ and $y = 1$. This is, in fact, the case.

First, $\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ is defined for all x because e^x and e^{-x} are both defined and positive for all real numbers x, so the denominator $e^x + e^{-x}$ is never 0. Thus the domain of $\tanh(x)$ is all of R, *i.e.* $(-\infty, \infty)$.

Second, since e^x and e^{-x} are always positive, $|e^x - e^{-x}| < e^x + e^{-x}$ for all x, from which it follows that $-1 < \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} < 1$. This isn't quite enough to conclude that the range is $(-1,1)$, though. For that we also need to combine the observations that $\tanh(x)$ is continuous and that:

$$\begin{aligned}
\lim_{x\to-\infty}\tanh(x) &= \lim_{x\to-\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}} = \lim_{x\to-\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}\cdot\frac{e^x}{e^x} = \lim_{x\to-\infty}\frac{(e^x)^2-1}{(e^x)^2+1} \\
&= \frac{0-1}{0+1} = -1 \quad [\text{since } e^x\to 0 \text{ as } x\to-\infty] \\
\lim_{x\to\infty}\tanh(x) &= \lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}} = \lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}\cdot\frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty}\frac{1-(e^{-x})^2}{1+(e^{-x})^2} \\
&= \frac{1-0}{1+0} = 1 \quad [\text{since } e^{-x}\to 0 \text{ as } x\to\infty]
\end{aligned}$$

Thus the range of $\tanh(x)$ is indeed $(-1,1)$, that is, all the real numbers y such that $-1<y<1$. ■

3. Find a formula for $\operatorname{arctanh}(x)$, the inverse function of $\tanh(x)$, by hand. What are the domain and range of $\operatorname{arctanh}(x)$? *[4]*

Solution. $y=\operatorname{arctanh}(x) \iff x=\tanh(y)$ because they are inverse functions, so we will find a formula for $\operatorname{arctanh}(x)$ by solving the equation $x=\tanh(y)$ for y.

$$\begin{aligned}
x=\tanh(y) &\iff x=\frac{e^y-e^{-y}}{e^y+e^{-y}} \iff x\left(e^y+e^{-y}\right)=e^y-e^-y \\
&\iff xe^y+xe^{-y}-e^y+e^{-y}=0 \iff: (x-1)e^y+(x+1)e^{-y}=0 \\
&\iff (x-1)e^ye^y+(x+1)e^{-y}e^y=0e^y=0 \iff (x-1)\left(e^y\right)^2+(x+1)=0 \\
&\iff \left(e^y\right)^2=-\frac{x+1}{x-1}=\frac{1+x}{1-x} \iff e^y=\sqrt{\frac{1+x}{1-x}}=\left(\frac{1+x}{1-x}\right)^{1/2} \\
&\iff y=\ln\left(\left(\frac{1+x}{1-x}\right)^{1/2}\right)=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)=\frac{\ln(1+x)-\ln(1-x)}{2}
\end{aligned}$$

Thus $\operatorname{arctanh}(x)=\dfrac{\ln(1+x)-\ln(1-x)}{2}$. Note that when we took the square root above, we didn't have to deal with the negative square root because $e^y>0$ for all real y.

Since $\ln(t)$ is defined for all positive real numbers t, $\ln(1+x)$ is defined for all $x>-1$ and $\ln(1-x)$ is defined for all $x<1$. It follows that $\operatorname{arctanh}(x)=\dfrac{\ln(1+x)-\ln(1-x)}{2}$ is defined for all x with $-1<x<1$, *i.e.* the domain of $\operatorname{arctanh}(x)$ is the interval $(-1,1)$.

For the range, observe, using the substitutions $t=1+x$ and $s=1-x$, that:

$$\begin{aligned}
\lim_{x\to-1^+}\operatorname{arctanh}(x) &= \lim_{x\to-1^+}\frac{\ln(1+x)-\ln(1-x)}{2} = \lim_{t\to0^+}\frac{\ln(t)-\ln(2-t)}{2} \\
&= -\infty \quad \text{since } \ln(t)\to-\infty \text{ and } \ln(2-t)\to\ln(2) \text{ as } t\to0^+ \\
\lim_{x\to1^-}\operatorname{arctanh}(x) &= \lim_{x\to1^-}\frac{\ln(1+x)-\ln(1-x)}{2} = \lim_{s\to0^+}\frac{\ln(2-s)-\ln(s)}{2} \\
&= +\infty \quad \text{since } \ln(2-s)\to\ln(2) \text{ and } \ln(s)\to-\infty \text{ as } s\to0^+
\end{aligned}$$

(Note that $-(-\infty) = +\infty$.) Since $\operatorname{arctanh}(x)$ is continuous where it is defined, being a composition of continuous functions, it follows that the range of $\operatorname{arctanh}(x)$ is all of $\mathbb{R}$, *i.e.* the range of $\operatorname{arctanh}(x)$ is the interval $(-\infty, \infty)$. ■

4. Use `Maple` to find a formula for $\operatorname{arctanh}(x)$. *[1]*

SOLUTION. Telling `Maple` to `solve(x=tanh(y),y)` gave something that was hard to interpret – hate it when *Root_Of* stuff shows up in an answer! – so the hard way it was:

> `solve(x = (exp(y)-exp(-y))/(exp(y)+exp(-y)), y)`

$$\ln\left(\frac{\sqrt{-(x-1)(1+x)}}{x-1}\right),\ \ln\left(-\frac{\sqrt{-(x-1)(1+x)}}{x-1}\right)$$

A little algebra will show that one of these is the answer obtained in the solution to **3**. [What about the other one?] ■

5. Find the derivative of $\operatorname{arctanh}(x)$ by hand, and then by using `Maple`. How does it compare to the derivative of $\arctan(x)$? *[3]*

SOLUTION. First, with `Maple`, taking advantage of the fact that it knows $\operatorname{arctanh}(x)$ so as not to have to type in the formula obtained in **3** and **4**:

> `diff(arctanh(x),x)`

$$\frac{1}{1-x^2}$$

Second, by hand:

$$\begin{aligned}
\frac{d}{dx}\operatorname{arctanh}(x) &= \frac{d}{dx}\left(\frac{\ln(1+x)-\ln(1-x)}{2}\right) = \frac{1}{2}\left(\frac{d}{dx}\ln(1+x) - \frac{d}{dx}\ln(1-x)\right) \\
&= \frac{1}{2}\left(\frac{1}{1+x}\cdot\frac{d}{dx}(1+x) - \frac{1}{1-x}\cdot\frac{d}{dx}(1-x)\right) \\
&= \frac{1}{2}\left(\frac{1}{1+x}\cdot 1 - \frac{1}{1-x}\cdot(-1)\right) = \frac{1}{2}\left(\frac{1}{1+x}+\frac{1}{1-x}\right) \\
&= \frac{1}{2}\cdot\frac{1\cdot(1-x)+1\cdot(1+x)}{(1+x)(1-x)} = \frac{1}{2}\cdot\frac{2}{1-x^2} = \frac{1}{1-x^2}
\end{aligned}$$ ■