Mathematics 1101Y – Calculus I: Functions and calculus of one variable
TRENT UNIVERSITY, 2013–2014

Solutions to Assignment #1
Plotting with `Maple`[†]

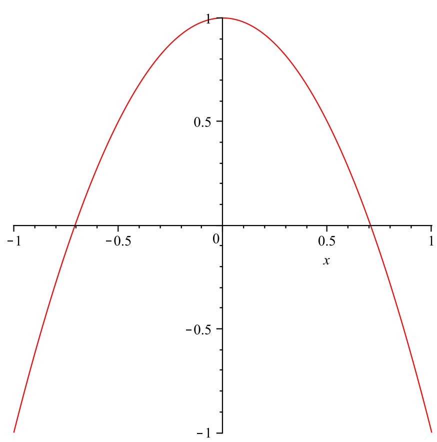
Before attempting the questions below, please read through Chapter 1 of the text for the basics of graphing various functions in Cartesian coordinates, and through §11.1 and §11.3 of Chapter 11 for the basics of parametric curves and polar coordinates, respectively. Some basics of graphing in Cartesian coordinates, using trigonometric functions, and so on, can also be found in the Academic Skills pamphlet *Formula for Success*. The basic definitions of how parametric curves and polar coordinates work are given in this assignment for your convenience, but you might want some additional explanations and examples. You should also read the handout *A very quick start with* `Maple` and play around with `Maple` a little. It might also be useful to skim though *Getting started with* `Maple 10` by Gilberto E. Urroz – read those parts concerned with plotting curves more closely! – and perhaps keep it handy as a reference. You can find links to these documents on the MATH 1101Y web page. `Maple`'s help facility may also come in handy, especially when trying to make out the intricacies of what the `plot` command and its options and variations do. Make use of the `Maple` labs and the instructor, too! Finally, don't forget that while ou may work together and look stuff up for the assignments. but you should write and/or type up what you submit by yourself.

A curve is easy to graph, at least in principle, if it can be described by a function of x in Cartesian coordinates.

1. Use `Maple` to plot the curves defined by $y = 1-2x^2$, $y = \sqrt{3}$, and $y = \sqrt{1-x^2}$, respectively, for $-1 \le x \le 1$ in each case. [Please submit a printout of your worksheet(s).] *[1.5]*

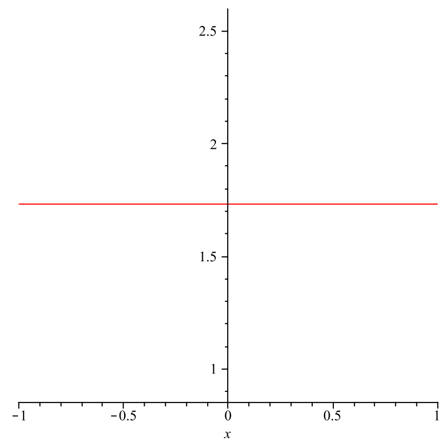
SOLUTION. Suitable instances of the `plot` command and their output are given below. [The graphs, both in this and the later solutions, have been reduced in size to save some space, but are otherwise unaltered.]

```
> plot(1-2*x^2, x = -1..1)
```

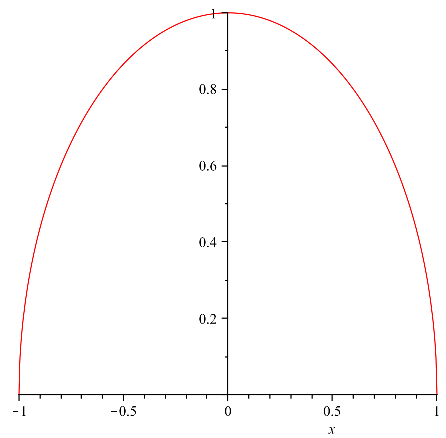


† . . . or is `Maple` plotting *against* you?

> plot(sqrt(3), x = -1..1)



> plot(sqrt(1-x^2), x = -1..1)



■

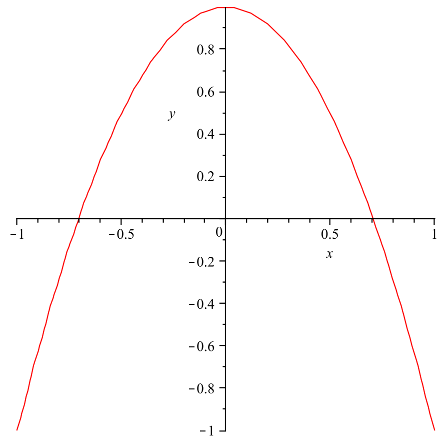
In many cases, a curve is difficult to break up into pieces that are defined by functions of x (or of y) and so is defined implicitly by an equation relating x and y; that is, the curve consists of all points (x, y) such that x and y satisfy the equation. One can, of course, also use implicit definitions to describe curves that can also be defined as the graphs of functions.

2. Use `Maple` to plot the curves implicitly defined by $y^2+4yx^2+4x^4 = 1$, for $-1 \leq y \leq 1$, $x^2+y^2 = 1$ for $y \geq 0$, $x^3+y^3-3xy = 0$, and $144\left(x^4+y^4\right)-225\left(x^2+y^2\right)+350x^2y^2+81 = 0$, respectively, the latter two for all x and y satisfying each equation. [Please submit a printout of your worksheet(s).] *[2]*

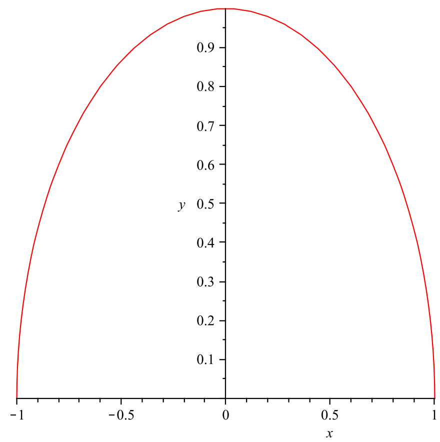
SOLUTION. To use the `implicitplot` command, as well as `polarplot` in the solution to **4**, we first need to load the `plots` package:

> with(plots)

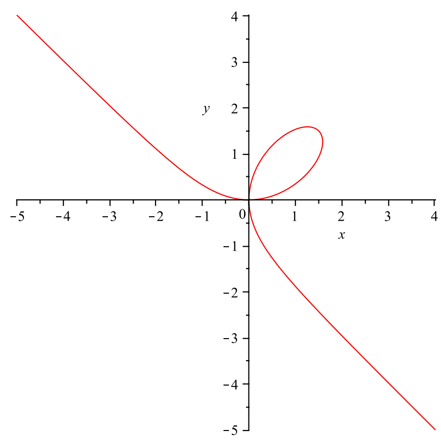
```
> implicitplot(y^2+4*y*x^2+4*x^4 = 1, x = -1..1, y = -1..1)
```




```
> implicitplot(x^2+y^2 = 1, x = -1..1, y = 0..1)
```

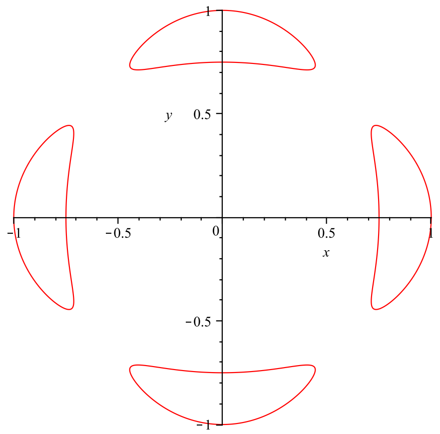



```
> implicitplot(x^3+y^3-3*x*y = 0, x = -5..5, y = -5..5, gridrefine = 4)
```



Note the use of the `gridrefine` option above, as well as below, to make the graph smoother. I played a bit with the limits for x and y in both to make sure that all of (the significant parts of) each curve were displayed.

> `implicitplot(144*(x^4+y^4)-225*(x^2+y^2)+350*x^2*y^2+81 = 0, x = -1..1, y = -1..1, gridrefine = 4)`



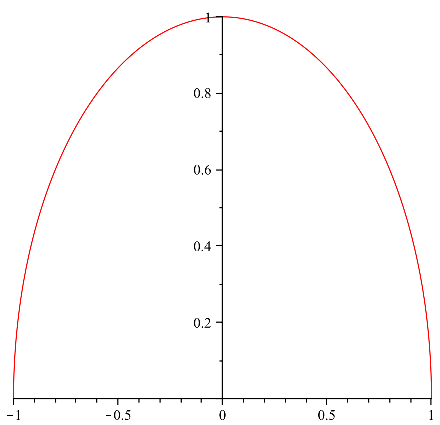
NOTE: This curve is studied in *A quartic with 28 real bitangents*, by R. Blum and A.P. Guinand, *Canadian Mathematical Bulletin*, vol. **7**, no. **3**, pp. 399-404, July 1964. Trent University was founded in the year this paper was published; Andre-Paul Guinand (1912–1987) was the first chair of Trent's Department of Mathematics. ■

Another way to describe or define a curve in two dimensions is by way of *parametric equations*, $x = f(t)$ and $y = g(t)$, where the x and y coordinates of points on the curve are simultaneously specified by plugging a third variable, called the *parameter* (in this case t), into functions $f(t)$ and $g(t)$. This approach can come in handy for situations where it is impossible to describe all of a curve as the graph of a function of x (or of y) and arises pretty naturally in various physics problems. (Think of specifying, say, the position (x, y) of a moving particle at time t.)

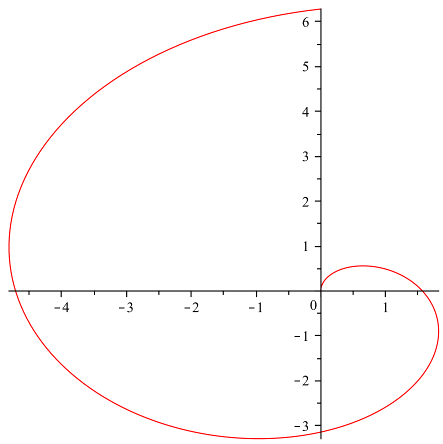
3. Use `Maple` to plot the parametric curves given by $x = \cos(t)$ and $y = \sin(t)$ for $0 \leq t \leq \pi$, $x = t\sin(t)$ and $y = t\cos(t)$ for $0 \leq t \leq 2\pi$, and $x = \dfrac{3t}{1+t^3}$ and $y = \dfrac{3t^2}{1+t^3}$ for all t which make sense. [Please submit a printout of your worksheet(s).] *[1.5]*

SOLUTION. For this one, we learn new tricks for using the `plot` command.

> `plot([cos(t), sin(t), t = 0..Pi])`

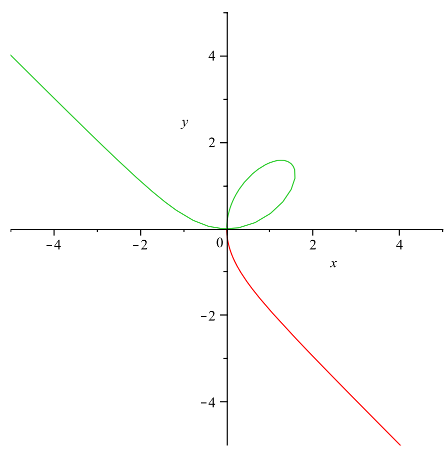


```
> plot([t*sin(t), t*cos(t), t = 0..2*Pi])
```



For the next one, we show off how to plot two parametric curves – well, the same one for different ranges of the parameter in this case – on the same graph.

```
> plot([[3*t/(1+t^3), 3*t^2/(1+t^3), t = -100..-1.1], [3*t/(1+t^3),
    3*t^2/(1+t^3), t = -.9..100]], x = -5..5, y = -5..5)
```

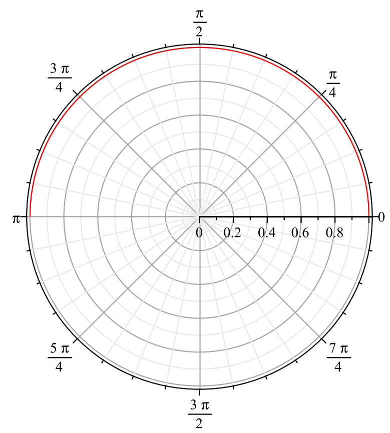


Polar coordinates are an alternative to the usual two-dimensional Cartesian coordinates. The idea is to locate a point by its distance r from the origin and its direction, which is given by the (counterclockwise) angle θ between the positive x-axis and the line from the origin to the point. Thus, if (r, θ) are the polar coordinates of some point, then its Cartesian coordinates are given by $x = r\cos(\theta)$ and $y = r\sin(\theta)$. (Note that for purposes of calculus it is usually more convenient to measure angles in radians rather than degrees.) Polar coordinates come in particularly handy when dealing with curves that wind around the origin, since such curves can often be conveniently represented by an equation of the form $r = f(\theta)$ for some function f of θ. If r is negative for a given θ, we interpret that as a distance of $|r|$ in the *opposite* direction, *i.e.* the direction $\theta + \pi$.

4. Use `Maple` to plot the curves in polar coordinates given by $r = 1$ for $0 \le \theta \le \pi$, $r = \dfrac{\sqrt{3}}{\sin(\theta)}$ for $\dfrac{2\pi}{3} \le \theta \le \dfrac{4\pi}{3}$, $r = \theta$ for $0 \le \theta \le 2\pi$, and $r = 1 + \cos(\theta)$ for $0 \le \theta \le 2\pi$, respectively. [Please submit a printout of your worksheet(s).] *[2]*

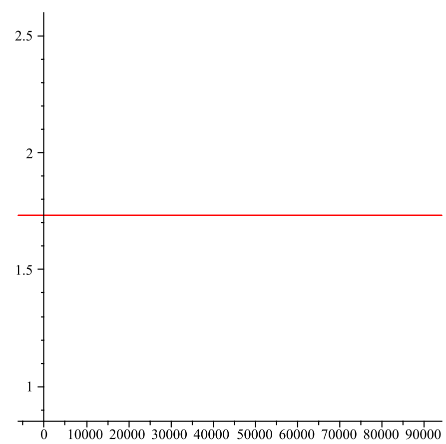
SOLUTION. One of the additional commands in the `plots` package, which we loaded to get `implicitplot` in the solutions for **2**, is `polarplot`.

```
> polarplot(1, theta = 0..Pi)
```



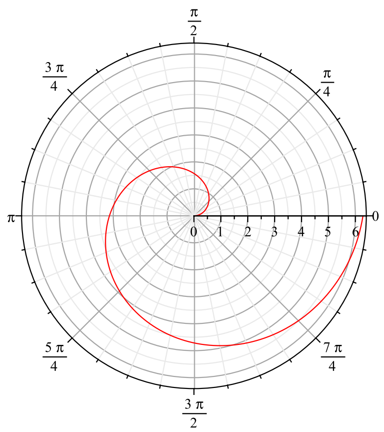
A quick-and-dirty alternative to `polarplot`, which has advantages in some situations, is the `coords = polar` option of the `plot` command, used in the next plot.

```
> plot(sqrt(3)/sin(theta), theta = 2*Pi*(1/3)..4*Pi*(1/3), coords = polar)
```

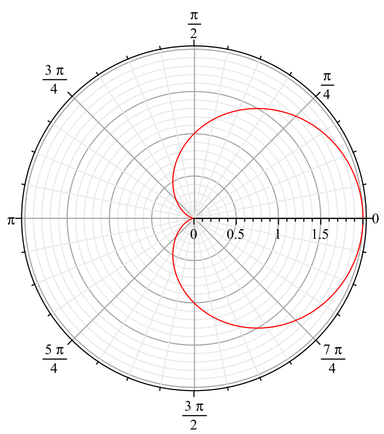


Now, back to `polarplot` . . .

```
> polarplot(theta, theta = 0..2*Pi)
```




```
> polarplot(1+cos(theta), theta = 0..2*Pi)
```



∎

5. Some of the curves in problems **1**–**4** are actually the same curve. (With different presentations!) Which ones are the same? *[2]*

SOLUTION. The following are the groups of curves that are the same:

- $y = 1 - 2x^2$, $-1 \le x \le 1$, from **1**, and $y^2 + 4yx^2 + 4x^4 = 1$, $-1 \le y \le 1$, from **2**.
- $y = \sqrt{1 - x^2}$, $-1 \le x \le 1$, from **1**, $x^2 + y^2 = 1$, $y \ge 0$, from **2**, $x = \cos(t)$ and $y = \sin(t)$, $0 \le t \le \pi$, from **3**, and $r = 1$, $0 \le \theta \le \pi$, from **4**.
- $x^3 + y^3 - 3xy = 0$, for all x and y satisfying the equation, from **2**, and $x = \dfrac{3t}{1+t^3}$ and $y = \dfrac{3t^2}{1+t^3}$, for all t which make sense [*i.e.* $t \ne -1$], from **3**.

There are also some near misses. First, $y = \sqrt{3}$, $-1 \le x \le 1$, from **1**, and $r = \dfrac{\sqrt{3}}{\sin(\theta)}$, $\dfrac{2\pi}{3} \le \theta \le \dfrac{4\pi}{3}$, from **4**, are different parts of the same straight line. Second, $x = t\sin(t)$

and $y = t\cos(t)$, $0 \le t \le 2\pi$, from **3**, and $r = \theta$, $0 \le \theta \le 2\pi$, from **4**, have the same shape, but each is a reflection of the other that is also rotated through a right angle. ■

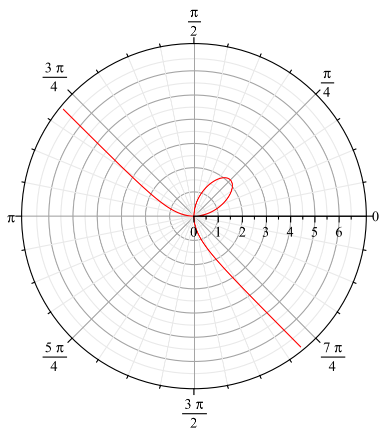
6. Find a representation of the form $r = f(\theta)$ in polar coordinates for the curve whose implicit equation in Cartesian coordinates is $x^3 + y^3 - 3xy = 0$, as best you can. *[1]*

SOLUTION. The idea is to plug the relations $x = r\cos(\theta)$ and $y = r\sin(\theta)$ into the given equation and try to solve for r.

$$\begin{aligned} x^3 + y^3 - 3xy = 0 &\implies r^3\cos^3(\theta) + r^3\sin^3(\theta) - 3r^2\cos(\theta)\sin(\theta) = 0 \\ &\implies r^2\left[r\cos^3(\theta) + r\sin^3(\theta) - 3\cos(\theta)\sin(\theta)\right] = 0 \\ &\implies r^2 = 0 \quad \text{or} \quad r\cos^3(\theta) + r\sin^3(\theta) - 3\cos(\theta)\sin(\theta) = 0 \\ &\implies r = 0 \quad \text{or} \quad r = \frac{3\cos(\theta)\sin(\theta)}{\cos^3(\theta) + \sin^3(\theta)} \end{aligned}$$

$r = 0$ just gives the origin, which $r = \dfrac{3\cos(\theta)\sin(\theta)}{\cos^3(\theta) + \sin^3(\theta)}$ also does (for example, when $\theta = 0$), so we really only have to check that $r = \dfrac{3\cos(\theta)\sin(\theta)}{\cos^3(\theta) + \sin^3(\theta)}$ is indeed the same curve as $x^3 + y^3 - 3xy = 0$. A quick and dirty way to check is to graph the two and compare. $x^3 + y^3 - 3xy = 0$ was plotted in solving **2**; for the other, we put `Maple` to work one more time:

```
> polarplot(3*cos(theta)*sin(theta)/(cos(theta)^3+sin(theta)^3),
  theta = -(1/4)*Pi+.1..3*Pi*(1/4)-.1)
```



It took a bit of tinkering with the values of θ, but they look the same to me! ■

Last modified 2013.10.13.