Mathematics 1101Y – Calculus I: functions and calculus of one variable
TRENT UNIVERSITY, 2013–2014

Solutions to the Quizzess

Quix #0. Monday, 9 September, 2013. *[10 minutes]*

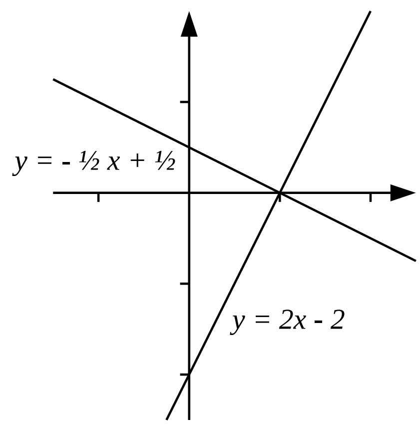
1. Find the x- and y-intercepts of the line given by $y = -\frac{1}{2}x + \frac{1}{2}$. *[1]*
2. Find the equation of the line which is perpendicular to the line given by $y = -\frac{1}{2}x + \frac{1}{2}$ and which has the same x-intercept. *[2]*
3. Sketch a graph of the line given by $y = -\frac{1}{2}x + \frac{1}{2}$ and the one that you obtained in answering question 2 above. *[2]*

SOLUTION TO 1. To obtain the y-intercept we set $x = 0$ and solve for y: $y = -\frac{1}{2} \cdot 0 + \frac{1}{2} = \frac{1}{2}$. For the x-intercept we set $y = 0$ and solve for x:

$$0 = -\frac{1}{2}x + \frac{1}{2} \implies \frac{1}{2}x = \frac{1}{2} \implies x = \frac{\frac{1}{2}}{\frac{1}{2}} = 1 \qquad \square$$

SOLUTION TO 2. The line given by the equation $y = mx + b$ will be perpendicular to the line given by $y = -\frac{1}{2}x + \frac{1}{2}$ exactly when $m = -\frac{1}{-\frac{1}{2}} = -(-2) = 2$. Now, the line given by $y = 2x + b$ has x-intercept 1 if $y = 2 \cdot 1 + b = 0$, *i.e.* if $b = -2$. Thus the equation of the line perpendicular to the line given by $y = -\frac{1}{2}x + \frac{1}{2}$ and which has the same x-intercept is $y = 2x - 2$. $\square$

SOLUTION TO 3. Here's a sketch of the two lines:



■

Quiz #1. Monday, 16 September, 2013. *[10 minutes]*

Do *one* (1) of questions 1 or 2.

1. Find the x- and y-intercepts, and the coordinates of the vertex, of the parabola $y = 2x^2 + 4x$. *[5]*
2. Verify that $\sec(x) - \tan(x) = \dfrac{1}{\sec(x) + \tan(x)}$. *[5]*

SOLUTION TO 1. To find the y-intercept, we set $x = 0$: $y = 2 \cdot 0^2 + 4 \cdot 0 = 0$.

To find the x-intercept, we set $y = 0$:

$$2x^2 + 4x = 0 \implies 2x(x+2) = 0 \implies x = 0 \text{ or } x = -2\,.$$

To find the location of the vertex, we complete the square:

$$y = 2x^2 + 4x = 2\left[x^2 + 2x\right] = 2\left[x^2 + 2x + 1 - 1\right] = 2\left[(x+1)^2 - 1^2\right] = 2(x+1)^2 - 2$$

It follows that the vertex is at the point that $x + 1 = 0$, namely $x = 1$; at which point $y = 2(-1+1)^2 - 2 = 2 \cdot 0^2 - 2 = -2$.

Thus, the x-intercepts of the parabola are -2 and 0, the y-intercept is 0, and the vertex is at $(-1, -2)$. ■

SOLUTION TO 2. One way to get the job done is to start with the indentity $1 + \tan^2(x) = \sec^2(x)$ and rearrange it:

$$\begin{aligned}\sec^2(x) = 1 + \tan^2(x) &\implies \sec^2(x) - \tan^2(x) = 1\\ &\implies (\sec(x) - \tan(x))\,(\sec(x) + \tan(x)) = 1\\ &\implies \sec(x) - \tan(x) = \frac{1}{\sec(x) + \tan(x)} \qquad \blacksquare\end{aligned}$$

Quiz #2. Monday, 23 September, 2013. *[10 minutes]*

1. Let $f(x) = \tan\left(x^2\right)$, where $0 \le x \le \sqrt{\pi/4}$. Find $f^{-1}(x)$. *[5]*

SOLUTION. Note that if $0 \le x \le \sqrt{\pi/4}$, then $0 \le x^2 \le \frac{\pi}{4}$, and so $0 \le \tan\left(x^2\right) \le 1$, so our solution for $f^{-1}(x)$ would need to work for $0 \le x \le 1$. Setting $x = f(y)$ and solving for y, as usual, gives:

$$x = \tan\left(y^2\right) \iff y^2 = \arctan(x) \iff y = \sqrt{\arctan(x)}$$

Thus $f^{-1}(x) = \sqrt{\arctan(x)}$.

CHECK: $f^{-1}(f(x)) = \sqrt{\arctan\left(\tan\left(x^2\right)\right)} = \sqrt{x^2} = x$, since $x \ge 0$ on the given domain of $f(x)$. Note also that when $0 \le x \le 1$, $0 \le \arctan(x) \le \frac{\pi}{4}$, so $f^{-1}(x) = \sqrt{\arctan(x)}$ is properly defined and its range, $\left[0, \sqrt{\pi/4}\right]$, matches the given domain of $f(x)$. ■

Quiz #3. Monday, 30 September, 2013. *[10 minutes]*

1. Compute $\lim_{x \to 0} \dfrac{(x+1)\sin(x)}{x^3 - x}$. *[5]*

Solution. Note that as $x \to 0$, $(x+1)\sin(x) \to (0+1)\sin(0) = 1 \cdot 0 = 0$ and $x^3 - x \to 0^3 - 0 = 0$. Since $\frac{0}{0}$ is undefined, one cannot, therefore, simply evaluate the expression in the given limit at $x = 0$ to compute the limit. We will compute it with a little help from algebra and the limit laws. Recall from class that $\lim_{x \to 0} \dfrac{\sin(x)}{x} = 0$.

$$\begin{aligned}\lim_{x \to 0} \frac{(x+1)\sin(x)}{x^3 - x} &= \lim_{x \to 0} \frac{(x+1)\sin(x)}{x\left(x^2 - 1\right)} = \lim_{x \to 0} \frac{x+1}{x^2-1} \cdot \frac{\sin(x)}{x} \\ &= \left[\lim_{x \to 0} \frac{x+1}{x^2-1}\right] \cdot \left[\lim_{x \to 0} \frac{\sin(x)}{x}\right] = \frac{0+1}{0^2-1} \cdot 1 = \frac{1}{-1} \cdot 1 = -1 \qquad \blacksquare\end{aligned}$$

Quiz #4. Monday, 7 October, 2013. *[10 minutes]*

1. Use the limit definition of the derivative to compute $f'(x)$ if $f(x) = 2x^2 - 7x$. *[4]*
2. Compute $f'(x)$ in a more sensible way. *[1]*

Solution to 1. Here goes:

$$\begin{aligned}f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left[2(x+h)^2 - 7(x+h)\right] - \left[2x^2 - 7x\right]}{h} \\ &= \lim_{h \to 0} \frac{\left[2\left(x^2 + 2hx + h^2\right) - 7x - 7h\right] - 2x^2 + 7x}{h} \\ &= \lim_{h \to 0} \frac{2x^2 + 4hx + 2h^2 - 7x - 7h - 2x^2 + 7x}{h} = \lim_{h \to 0} \frac{4hx + 2h^2 - 7h}{h} \\ &= \lim_{h \to 0} \frac{h\left(4x + 2h - 7\right)}{h} = \lim_{h \to 0} \left(4x + 2h - 7\right) = 4x + 2 \cdot 0 - 7 = 4x - 7 \qquad \blacksquare\end{aligned}$$

Solution to 2. Using the Power, multiplication by constants, and Sum rules for derivatives:

$$f'(x) = \frac{d}{dx}\left(2x^2 - 7x\right) = 2\left[\frac{d}{dx}x^2\right] - 7\left[\frac{d}{dx}x\right] = 2 \cdot 2x - 7 \cdot 1 = 4x - 7 \qquad \blacksquare$$

Quiz #5. Tuesday, 15 October, 2013. *[10 minutes]*

1. Compute $f'(x)$ if $f(x) = \left(1+x^2\right)\arctan(x)$. *[2]*
2. Compute $g'(x)$ if $g(x) = \cos\left(\sqrt{1+e^x}\right)$. *[3]*

SOLUTION TO 1. Product and Power Rules, mainly:

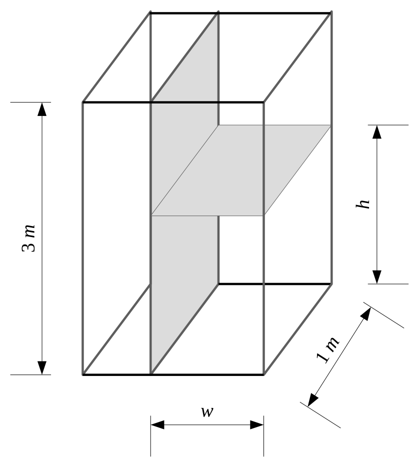
$$f'(x) = \frac{d}{dx}\left[\left(1+x^2\right)\arctan(x)\right] = \left[\frac{d}{dx}\left(1+x^2\right)\right]\arctan(x) + \left(1+x^2\right)\left[\frac{d}{dx}\arctan(x)\right]$$
$$= (0+2x)\arctan(x) + \left(1+x^2\right)\frac{1}{1+x^2} = 2x\arctan(x) + 1 \qquad \blacksquare$$

SOLUTION TO 2. Lots of Chain Rule and a bit of Power Rule:

$$g'(x) = \frac{d}{dx}\cos\left(\sqrt{1+e^x}\right) = -\sin\left(\sqrt{1+e^x}\right)\left[\frac{d}{dx}\sqrt{1+e^x}\right]$$
$$= -\sin\left(\sqrt{1+e^x}\right)\frac{1}{2\sqrt{1+e^x}}\left[\frac{d}{dx}\left(1+e^x\right)\right]$$
$$= \frac{-\sin\left(\sqrt{1+e^x}\right)}{2\sqrt{1+e^x}}\left(0+e^x\right) = \frac{-e^x\sin\left(\sqrt{1+e^x}\right)}{2\sqrt{1+e^x}}$$

If you blinked and missed the use of the Power Rule above, it plays a part because $\sqrt{1+e^x} = (1+e^x)^{1/2}$. $\blacksquare$

Quiz #6. Monday, 28 October, 2013. *[15 minutes]*



The interior of a trash compactor has the shape of a rectangular box 3 m high, 1 m long, and w m wide. The width w diminishes at a rate of 1 m/min when the trash is compacted. The compactor is turned on with nothing but 6 m^3 of water inside it. No water leaks out (until it overflows :-), and you may assume that the water does not slosh about, the surface remaining perfectly level throughout.

1. How is the height of the water in the trash compactor changing at the instant that $w = 3$ m? *[5]*

SOLUTION. The volume of the water in the tank doesn't change: it remains 6 m^3 until h exceeds 3 m. At any given instant, however, it occupies a rectangular shape with height h m, width w m, and length 1 m, which has volume $V = hw1 = hw$ m^3. Thus $6 = hw$. It follows that $0 = \frac{d}{dt}6 = \frac{d}{dt}(hw) = \frac{dh}{dt}\cdot w + h\cdot\frac{dw}{dt}$, and so at any given instant $\frac{dh}{dt} = -\frac{h}{w}\cdot\frac{dw}{dt}$. When $w = 1$ m, $h = \frac{6}{w} = \frac{6}{3} = 2$ m, so $\left.\frac{dh}{dt}\right|_{w=3} = -\frac{2}{3}\cdot(-1) = \frac{2}{3}$ m/min. That is, the height of the water in the trash compactor is increasing at a rate of $\frac{2}{3}$ m/min at the instant that $w = 3$ m. $\blacksquare$

Quiz #7. Monday, 4 November, 2013. *[20 minutes]*

1. Find the domain and any and all intercepts, vertical and horizontal asymptotes, intervals of increase and decrease, maxima and and minima, intervals of concavity, and inflection points, of $y = \dfrac{1-x^2}{1+x^2}$, and sketch the graph. *[5]*

SOLUTION. We'll run through the usual checklist:

i. Domain. $y = \frac{1-x^2}{1+x^2}$ is defined for all x that do not make the denominator 0. Since the denominator, $1+x^2$, is always at least 1, it is never 0, so the domain of $y = \frac{1-x^2}{1+x^2}$ is all real numbers x.

ii. Intercepts. When $x = 0$, $y = \frac{1-0^2}{1+0^2} = 1$, so the y-intercept is $y = 1$.

To get $y = 0$, we must have $1 - x^2 = 0$, *i.e.* $x^2 = 1$, which happens exactly when $x = \pm 1$. Thus there are two x-intercepts, $x = -1$ and $x = 1$.

iii. Vertical asymptotes. Since $y = \frac{1-x^2}{1+x^2}$ is defined and continuous for all x, it cannot have any vertical asymptotes.

iv. Horizontal asymptotes. Here we go:

$$\lim_{x\to-\infty} \frac{1-x^2}{1+x^2} = \lim_{x\to-\infty} \frac{1-x^2}{1+x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{\frac{1}{x^2}-1}{\frac{1}{x^2}+1} = \frac{0-1}{0+1} = -1 \quad \text{and}$$
$$\lim_{x\to+\infty} \frac{1-x^2}{1+x^2} = \lim_{x\to+\infty} \frac{1-x^2}{1+x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to+\infty} \frac{\frac{1}{x^2}-1}{\frac{1}{x^2}+1} = \frac{0-1}{0+1} = -1\,,$$

since $\frac{1}{x^2} \to 0$ as either $x \to -\infty$ or $x \to +\infty$. Thus $y = \frac{1-x^2}{1+x^2}$ has the horizontal asymptote $y = -1$ in both directions.

v. Increase, decrease, maxima, and minima. First,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) = \frac{\left[\frac{d}{dx}\left(1-x^2\right)\right]\cdot\left(1+x^2\right) - \left(1-x^2\right)\cdot\left[\frac{d}{dx}\left(1+x^2\right)\right]}{\left(1+x^2\right)^2}$$
$$= \frac{[-2x]\cdot\left(1+x^2\right) - \left(1-x^2\right)\cdot[2x]}{\left(1+x^2\right)^2} = \frac{-2x-2x^3-2x+2x^3}{\left(1+x^2\right)^2} = \frac{-4x}{\left(1+x^2\right)^2}\,.$$

Second, note that $\frac{dy}{dx} = \frac{-4x}{(1+x^2)^2} = 0$ exactly when $-4x = 0$, *i.e.* when $x = 0$, and that the denominator is always ≥ 1, and hence always positive. It follows that $\frac{dy}{dx} > 0$ exactly when $-4x > 0$, *i.e.* when $x < 0$, and that $\frac{dy}{dx} < 0$ exactly when $-4x < 0$, *i.e.* when $x > 0$. Thus y is increasing when $x < 0$ and decreasing when $x > 0$, and hence has a maximum at $x = 0$.

We summarise this in the usual table:

x	$(-\infty, 0)$	0	$(0, \infty)$
$\frac{dy}{dx}$	$+$	0	$-$
y	$\uparrow$	max	$\downarrow$

vi. Concavity and points of inflection. First,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-4x}{\left(1+x^2\right)^2}\right) = \frac{\left[\frac{d}{dx}(-4x)\right]\cdot\left(1+x^2\right)^2 - (-4x)\cdot\left[\frac{d}{dx}\left(1+x^2\right)^2\right]}{\left(\left(1+x^2\right)^2\right)^2}$$

$$= \frac{[-4]\cdot\left(1+x^2\right)^2 + 4x\cdot\left[2\left(1+x^2\right)\cdot\frac{d}{dx}\left(1+x^2\right)\right]}{\left(1+x^2\right)^4}$$

$$= \frac{\left(1+x^2\right)\left(-4\cdot\left(1+x^2\right)+4x\cdot 2\cdot 2x\right)}{\left(1+x^2\right)^4} = \frac{-4\cdot\left(1+x^2\right)+4x\cdot 2\cdot 2x}{\left(1+x^2\right)^3}$$

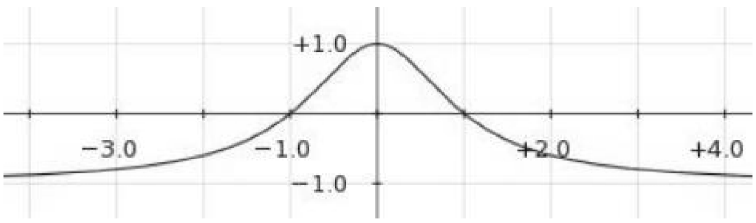
$$= \frac{-4-4x^2+8x^2}{\left(1+x^2\right)^3} = \frac{4x^2-4}{\left(1+x^2\right)^3} = \frac{4\left(x^2-1\right)}{\left(1+x^2\right)^3}.$$

Second, note that $\frac{d^2y}{dx^2} = \frac{4\left(x^2-1\right)}{\left(1+x^2\right)^3} = 0$ exactly when $x^2-1=0$, *i.e.* when $x=\pm 1$. Moreover, since $\left(1+x^2\right)^3 \geq 1 > 0$ for all x, $\frac{d^2y}{dx^2}$ is positive or negative exactly when x^2-1 is positive or negative. It follows that $\frac{d^2y}{dx^2} > 0$ when $x^2-1>0$, *i.e.* when $|x|>1$, and $\frac{d^2y}{dx^2} < 0$ when $x^2-1<0$, *i.e.* when $|x|<1$. Thus $y=\frac{1-x^2}{1+x^2}$ is concave down for $-1<x<1$ and concave up for $x<-1$ and for $x>1$. This means that there are points of inflection at $x=-1$ and $x=1$.

We summarise this in the usual table:

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
$\frac{d^2y}{dx^2}$	$+$	0	$-$	0	$+$
y	$\smile$	inflection point	$\frown$	inflection point	$\smile$

vii. The sketch. Cheating slightly, I used `KAlgebra` to plot the graph:



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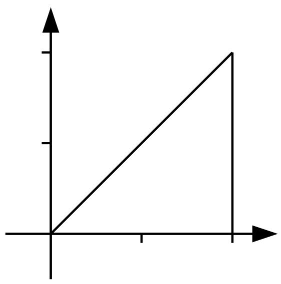
Quiz #8. Monday, ~~18~~ 25 November, 2013.

1. Sketch the region whose area is computed by $\int_0^2 x\,dx$ and use its shape to find the area. *[5]*

SOLUTION. Here's a crude sketch:



The region is a triangle with base and height 2, and so has area $\frac{1}{2} \cdot 2 \cdot 2 = 2$. ■

Quiz #9. Monday, 25 November, 2013.

1. Use the Left-hand or the Right-hand Rule to compute $\int_0^2 x\,dx$. *[5]*

SOLUTION. *i. Left-hand Rule.* We plug things into the Left-hand Rule formula

$$\int_0^2 x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[f\left(a + (i-1)\frac{b-a}{n}\right)\right] \cdot \left[\frac{b-a}{n}\right]$$

and go. Note that in this case $a = 0$, $b = 2$, and $f(x) = x$:

$$\begin{aligned}
\int_0^2 x\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[0 + (i-1)\frac{2-0}{n}\right] \cdot \left[\frac{2-0}{n}\right] = \lim_{n\to\infty} \sum_{i=1}^{n} \left[(i-1)\frac{2}{n}\right] \cdot \left[\frac{2}{n}\right] \\
&= \lim_{n\to\infty} \left[\frac{2}{n}\right] \sum_{i=1}^{n} \left[(i-1)\frac{2}{n}\right] = \lim_{n\to\infty} \left[\frac{2}{n}\right] \left[\frac{2}{n} \sum_{i=1}^{n} (i-1)\right] \\
&= \lim_{n\to\infty} \frac{4}{n^2} \left[\left(\sum_{i=1}^{n} i\right) - \left(\sum_{i=1}^{n} 1\right)\right] = \lim_{n\to\infty} \frac{4}{n^2} \left[\frac{n(n+1)}{2} - n\right] \\
&= \lim_{n\to\infty} \frac{4}{n^2} \left[\frac{n^2}{2} + \frac{n}{2} - n\right] = \lim_{n\to\infty} \frac{4}{n^2} \left[\frac{n^2}{2} - \frac{n}{2}\right] = \lim_{n\to\infty} \left[\frac{4}{n^2} \cdot \frac{n^2}{2} - \frac{4}{n^2} \cdot \frac{n}{2}\right] \\
&= \lim_{n\to\infty} \left[2 - \frac{2}{n}\right] = 2 - 0 = 2\,,
\end{aligned}$$

since $\frac{2}{n} \to 0$ as $n \to \infty$. □

ii. Right-hand Rule. This is much the same, except that we use the Right-hand Rule formula:

$$\int_0^2 x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[f\left(a + i\frac{b-a}{n}\right)\right] \cdot \left[\frac{b-a}{n}\right]$$

We still have $a = 0$, $b = 2$, and $f(x) = x$:

$$\begin{aligned}\int_0^2 x\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\left[0 + i\frac{2-0}{n}\right]\cdot\left[\frac{2-0}{n}\right] = \lim_{n\to\infty}\sum_{i=1}^{n}\left[i\frac{2}{n}\right]\cdot\left[\frac{2}{n}\right] \\ &= \lim_{n\to\infty}\left[\frac{2}{n}\right]\sum_{i=1}^{n}\left[i\frac{2}{n}\right] = \lim_{n\to\infty}\left[\frac{2}{n}\right]\left[\frac{2}{n}\sum_{i=1}^{n} i\right] = \lim_{n\to\infty}\frac{4}{n^2}\left[\sum_{i=1}^{n} i\right] \\ &= \lim_{n\to\infty}\frac{4}{n^2}\left[\frac{n(n+1)}{2}\right] = \lim_{n\to\infty}\frac{4}{n^2}\left[\frac{n^2}{2}+\frac{n}{2}\right] = \lim_{n\to\infty}\left[\frac{4}{n^2}\cdot\frac{n^2}{2}+\frac{4}{n^2}\cdot\frac{n}{2}\right] \\ &= \lim_{n\to\infty}\left[2+\frac{2}{n}\right] = 2 - 0 = 2\,,\end{aligned}$$

since $\frac{2}{n} \to 0$ as $n \to \infty$. $\square$

Quiz #10. Monday, 2 December, 2013. *[10 minutes]*

1. Compute $\displaystyle\int_0^1 \frac{x}{1+x^2}\,dx$. *[5]*

Solution. We will use the substitution $u = 1 + x^2$, so $\frac{du}{dx} = \frac{d}{dx}\left(1+x^2\right) = 0 + 2x = 2x$, and thus $du = 2x\,dx$ and $\begin{matrix} x & u \\ 0 & 1 \\ 1 & 2 \end{matrix}$. Since we only have $x\,dx$ instead of $2x\,dx$ available to us in the integrand, we solve for what we do have: $x\,dx = \frac{1}{2}\,du$. Then

$$\begin{aligned}\int_0^1 \frac{x}{1+x^2}\,dx &= \int_1^2 \frac{1}{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\int_1^2 \frac{1}{u}\,du = \frac{1}{2}\ln(u)\Big|_1^2 \\ &= \frac{1}{2}\ln(2) - \frac{1}{2}\ln(1) = \frac{1}{2}\ln(2)\,,\end{aligned}$$

since $\ln(1) = 0$. $\blacksquare$

Quiz #11. Monday, 6 January, 2014. *[10 minutes]*

1. Compute $\displaystyle\int x^2 \ln(x)\,dx$. *[5]*

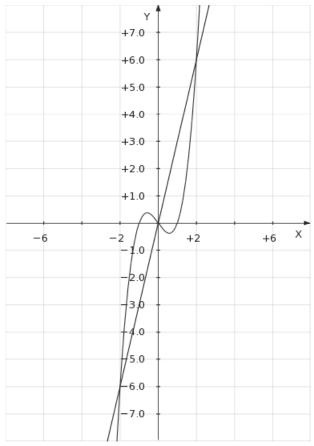
Solution. We will use integration by parts, with $u = \ln(x)$ and $v' = x^2$, so $u' = \frac{1}{x}$ and $v = \frac{1}{3}x^3$. Note that we are computing an indefinite integral.

$$\begin{aligned}\int x^2\ln(x)\,dx &= \int uv'\,dx = uv - \int u'v\,dx = \ln(x)\cdot\frac{1}{3}x^3 - \int \frac{1}{x}\cdot\frac{1}{3}x^3\,dx \\ &= \frac{1}{3}x^3\ln(x) - \frac{1}{3}\int x^2\,dx = \frac{1}{3}x^3\ln(x) - \frac{1}{3}\cdot\frac{1}{3}x^3 + C \\ &= \frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3 + C = \frac{1}{3}x^3\left(\ln(x) - \frac{1}{3}\right) + C \qquad \blacksquare\end{aligned}$$

Quiz #12. Monday, 13 January, 2014. *[12 minutes]*

1. Find the area of the region between $y = x^3 - x$ and $y = 3x$, for $-1 \leq x \leq 1$. *[5]*

SOLUTION. Instead of a quick-and-dirty sketch, here's a plot of the two curves generated by a program called `kmplot`:



One should be able to get a similar plot out of `Maple` with a command like:

> `plot([x,3x,x=-3..3],[x,x^3-x,x=-3..3])`

Even a quick-and-dirty sketch would (probably!) suggest that the line and the cubic intersect three times, one of those being at the origin. A little algebra will pin them down pretty quickly:

$$\begin{aligned} x^3 - x = y = 3x \quad &\Longrightarrow \quad x^3 = 4x \quad \Longrightarrow \quad x = 0 \text{ or } x^2 = 4 \\ &\Longrightarrow \quad x = 0 \text{ or } x = \pm\sqrt{4} = \pm 2 \end{aligned}$$

Note that the points, other than $x = 0$, where the curves intersect are beyond the limits of -1 to 1 specified for x, so we don't have to take them into account.

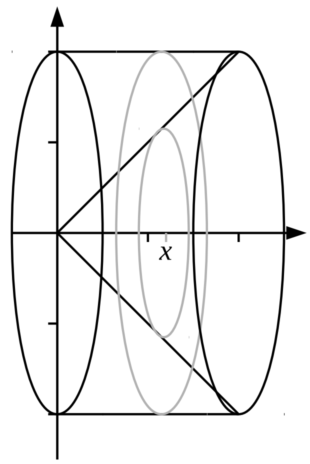
A quick glance at the plot above, or a not-too-awful sketch, makes it pretty clear that for $-1 \leq x \leq 0$, $y = x^3 - x$ is above $y = 3x$, while for $0 \leq x \leq 1$, $y = 3x$ is above $y = x^3 - x$. Thus the area of the region is:

$$\begin{aligned} A &= \int_{-1}^{0} \left(\left[x^3 - x\right] - 3x \right) \, dx + \int_{0}^{1} \left(3x - \left[x^3 - x\right] \right) \, dx \\ &= \int_{-1}^{0} \left(x^3 - 4x \right) \, dx + \int_{0}^{1} \left(4x - x^3 \right) \, dx \\ &= \left(\frac{x^4}{4} - 4\frac{x^2}{2} \right) \Bigg|_{-1}^{0} + \left(4\frac{x^2}{2} - \frac{x^4}{4} \right) \Bigg|_{0}^{1} \\ &= \left[0 - \left(\frac{1}{4} - 2 \right) \right] + \left[\left(2 - \frac{1}{4} \right) - 0 \right] \\ &= -\left(-\frac{7}{4} \right) + \frac{7}{4} = \frac{7}{4} + \frac{7}{4} = \frac{14}{4} = \frac{7}{2} = 3.5 \qquad \blacksquare \end{aligned}$$

Quiz #13. Monday, 20 January, 2014. *[15 minutes]*

1. Sketch the solid obtained by revolving the region below $y = 2$ and above $y = x$, for $0 \leq x \leq 2$, about the x-axis, and find its volume. *[5]*

SOLUTION. We'll use the disk/washer method to compute the volume of the solid. Here's a sketch of the solid, a cylinder with a cone cut out of it, and a generic washer at x:



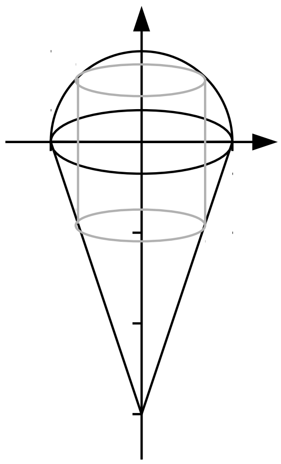
Note that the radius of the (outside of the) washer is $R = 2 - 0 = 2$, and that the radius of the hole in the washer is $r = y - 0 = y = x$. We are give that the region has $0 \leq x \leq 2$. Thus the volume of the solid is:

$$V = \int_0^2 \pi \left(R^2 - r^2\right)\, dx = \pi \int_0^2 \left(2^2 - x^2\right)\, dx = \pi \int_0^2 \left(4 - x^2\right)\, dx = \pi \left(4x - \frac{x^3}{3}\right)\Big|_0^2$$
$$= \pi \left(4 \cdot 2 - \frac{2^3}{3}\right) - \pi \left(4 \cdot 0 - \frac{0^3}{3}\right) = \pi \left(8 - \frac{8}{3}\right) - \pi \cdot 0 = \frac{24-8}{3}\pi - 0 = \frac{16}{3}\pi \qquad \blacksquare$$

Quiz #14. Monday, 27 January, 2014. *[20 minutes]*

1. Sketch the "ice-cream cone" solid obtained by revolving the region below $y = \sqrt{1-x^2}$ and above $y = 3x - 3$, for $0 \le x \le 1$, about the y-axis, and find its volume. *[5]*

SOLUTION. We'll use the cylindrical shell method to compute the volume of the ice-cream cone. Here's a sketch of this solid, a point-down cone with a hemisphere on top, and a generic cylindrical shell:



Note that the radius of the cylindrical shell that meets the x-axis at x is $R = x - 0 = x$ and its height is $h = \sqrt{1-x^2} - (3x-3)$. We are given that $0 \le x \le 1$ for our original region. Thus the volume of the solid is:

$$\begin{aligned}
V &= \int_0^1 2\pi Rh\,dx = 2\pi \int_0^1 x\left[\sqrt{1-x^2} - (3x-3)\right]\,dx \\
&= 2\pi \int_0^1 x\sqrt{1-x^2}\,dx - 2\pi \int_0^1 x(3x-3)\,dx \\
&= 2\pi \int_0^1 x\left(1-x^2\right)^{1/2}\,dx - 2\pi \int_0^1 \left(3x^2 - 3x\right)\,dx \\
&= \pi \int_1^0 u^{1/2}\,(-1)\,du - 2\pi \left(x^3 - \frac{3}{2}x^2\right)\Big|_1^0 \\
&= \pi \int_0^1 u^{1/2}\,du - 2\pi\left[\left(1^3 - \frac{3}{2}1^2\right) - \left(0^3 - \frac{3}{2}0^2\right)\right] = \pi \left.\frac{u^{3/2}}{3/2}\right|_0^1 - 2\pi\left[-\frac{1}{2} - 0\right] \\
&= \pi\left[\frac{2}{3}1^{3/2} - \frac{2}{3}0^{3/2}\right] + \pi = \pi\left[\frac{2}{3} - 0\right] + \pi = \frac{5}{3}\pi \qquad \blacksquare
\end{aligned}$$

We'll use the substitution $u = 1 - x^2$, so $du = -2x\,dx$, *i.e.* $(-1)\,du = 2x\,dx$, and $\begin{smallmatrix} x & 0 & 1 \\ u & 1 & 0 \end{smallmatrix}$, to do the first integral.

Quiz #15. Monday, 10 February, 2014. *[10 minutes]*

1. Compute $\displaystyle\int \frac{1}{\sqrt{9x^2+25}}\,dx$. *[5]*

SOLUTION. We will use the trigonometric substitution $x = \frac{5}{3}\tan(\theta)$, so $dx = \frac{5}{3}\sec^2(\theta)\,d\theta$. Anticipating that we will eventually have to substitute back, note that this also means that $\tan(\theta) = \frac{3}{5}x$ and $\sec(\theta) = \sqrt{\sec^2(\theta)} = \sqrt{\tan^2(\theta)+1} = \sqrt{\left(\frac{3}{5}x\right)^2+1} = \sqrt{\frac{9}{25}x^2+1}$. Here goes:

$$\begin{aligned}
\int \frac{1}{\sqrt{9x^2+25}}\,dx &= \int \frac{1}{\sqrt{9\left(\frac{5}{3}\tan(\theta)\right)^2+25}}\,\frac{5}{3}\sec^2(\theta)\,d\theta = \frac{5}{3}\int \frac{\sec^2(\theta)}{\sqrt{9\frac{25}{9}\tan^2(\theta)+25}}\,d\theta \\
&= \frac{5}{3}\int \frac{\sec^2(\theta)}{\sqrt{25\left(\tan^2(\theta)+1\right)}}\,d\theta = \frac{5}{3}\int \frac{\sec^2(\theta)}{5\sqrt{sec(\theta)}}\,d\theta = \frac{1}{3}\int \frac{\sec^2(\theta)}{\sec(\theta)}\,d\theta \\
&= \frac{1}{3}\int \sec(\theta)\,d\theta = \ln\left(\sec(\theta)+\tan(\theta)\right)+C \\
&= \ln\left(\sqrt{\frac{9}{25}x^2+1}+\frac{3}{5}x\right)+C = \ln\left(\frac{1}{5}\sqrt{9x^2+25}+\frac{3}{5}x\right)+C
\end{aligned}$$

That last simplification is really just gilding the lily ... ■

Quiz #16. Monday, 24 February, 2014. *[15 minutes]*

1. Compute $\displaystyle\int \frac{1}{x^3+x^2}\,dx$. *[5]*

SOLUTION. Since we are integrating a rational function, we will use partial fractions. Note that the denominator has higher degree than the numerator and that the denominator factors as $x^3+x^2 = x^2(x+1)$. Hence

$$\begin{aligned}
\frac{1}{x^3+x^2} &= \frac{1}{x^2(x+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1} = \frac{Ax(x+1)+B(x+1)+Cx^2}{x^2(x+1)} \\
&= \frac{(A+C)x^2+(A+B)x+B}{x^3+x^2}
\end{aligned}$$

for some unknown constants A, B, and C. Comparing numerators, we get that $A+C=0$, $A+B=0$, and $B=1$. It follows from the second and last of these equations that $A=-1$, and then from the first equation that $C=1$. Thus

$$\begin{aligned}
\int \frac{1}{x^3+x^2}\,dx &= \int \left(\frac{-1}{x}+\frac{1}{x^2}+\frac{1}{x+1}\right)dx = -\int \frac{1}{x}\,dx + \int x^{-2}\,dx + \int \frac{1}{x+1}\,dx \\
&= -\ln(x) - x^{-1} + \int \frac{1}{u}\,du \qquad \text{(Substituting } u = x+1\text{, so } du = dx\text{.)} \\
&= -\ln(x) - \frac{1}{x} + \ln(u) + K = -\ln(x) - \frac{1}{x} + \ln(x+1) + K\,.
\end{aligned}$$

(K was used for the generic constant because we had already used C for another purpose.) Perversely, if you mistakenly assume x^2 is an irreducible quadratics and set up the partial fractions as $\frac{Ax+B}{x^2}+\frac{C}{x+1}$, you still get $A=-1$ and $B=C=1$. ■

Quiz #17. Monday, 3 March, 2014. *[12 minutes]*

1. Compute $\displaystyle\int_0^\infty \frac{2x}{(1+x^2)^2}\,dx$. *[5]*

SOLUTION. Here goes:

$$\int_0^\infty \frac{2x}{(1+x^2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{2x}{(1+x^2)^2}\,dx$$

We'll use the substitution $u = 1+x^2$, so $du = 2x\,dx$ and $\begin{smallmatrix} x & 0 & t \\ u & 1 & 1+t^2 \end{smallmatrix}$.

$$= \lim_{t\to\infty}\int_1^{1+t^2}\frac{1}{u^2}\,du = \lim_{t\to\infty}\int_1^{1+t^2} u^{-2}\,du = \lim_{t\to\infty}\left.\frac{u^{-2+1}}{-2+1}\right|_1^{1+t^2}$$

$$= \lim_{t\to\infty}\left.\frac{u^{-1}}{-1}\right|_1^{1+t^2} = \lim_{t\to\infty}\left.\frac{-1}{u}\right|_1^{1+t^2} = \lim_{t\to\infty}\left(\frac{-1}{1+t^2} - \frac{-1}{1}\right)$$

$$= \lim_{t\to\infty}\left(1 - \frac{1}{1+t^2}\right) = 1 - 0 = 1\,,$$

since, as $t\to\infty$, we have $1+t^2\to\infty$ even faster, so $\dfrac{1}{1+t^2}\to 0$. ■

Quiz #18. Monday, 10 March, 2014. *[10 minutes]*

1. Compute $\displaystyle\lim_{n\to\infty}\frac{\ln(n^n)}{n^2}$. *[5]*

SOLUTION. Here goes:

$$\lim_{n\to\infty}\frac{\ln(n^n)}{n^2} = \lim_{n\to\infty}\frac{n\ln(n)}{n^2} = \lim_{n\to\infty}\frac{\ln(n)}{n} = \lim_{x\to\infty}\frac{\ln(x)}{x}\begin{matrix}\to\infty\\ \to\infty\end{matrix}$$

so, using l'Hôpital's Rule,

$$= \lim_{x\to\infty}\frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}x} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = \lim_{x\to\infty}\frac{1}{x}\begin{matrix}\to 1\\ \to\infty\end{matrix} = 0 \qquad ■$$

Quiz #19. Monday, 17 March, 2014. *[10 minutes]*

1. Determine whether the series $\displaystyle\sum_{n=1}^\infty \frac{1-\frac{1}{n}}{n^2}$ converges or diverges. *[5]*

SOLUTION. We will use the (Basic) Comparison Test. If $n\geq 1$, then $0 < \frac{1}{n} \leq 1$, so $0 \leq 1-\frac{1}{n} < 1$. It follows that for all $n\geq 1$, $0 \leq \dfrac{1-\frac{1}{n}}{n^2} < \dfrac{1}{n^2}$. Since $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2}$ converges (by the p-Test [since $p = 2 > 1$ here], or by the Integral Test, or from class …), it follows by the Comparison Test that $\displaystyle\sum_{n=1}^\infty \frac{1-\frac{1}{n}}{n^2}$ also converges. ■

Quiz #20. Monday, 24 March, 2014. *[10 minutes]*

1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+1)}{5^{n/2}}$ converges or diverges. *[5]*

SOLUTION. We will use the Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(n+1)((n+1)+1)}{5^{(n+1)/2}}}{\frac{n(n+1)}{5^{n/2}}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(n+2)}{5^{n/2+1/2}}\cdot\frac{5^{n/2}}{n(n+1)}\right| \\
&= \lim_{n\to\infty}\frac{(n+1)(n+2)}{n(n+1)}\cdot\frac{5^{n/2}}{5^{n/2+1/2}} = \lim_{n\to\infty}\frac{(n+2)}{n}\cdot\frac{5^{n/2}}{5^{n/2}5^{1/2}} \\
&= \lim_{n\to\infty}\left(1+\frac{2}{n}\right)\cdot\frac{1}{5^{1/2}} = (1+0)\cdot\frac{1}{5^{1/2}} = \frac{1}{5^{1/2}} = \frac{1}{\sqrt{5}}
\end{aligned}$$

Since $5 > 4$ and $\sqrt{x}$ is an increasing function, $\sqrt{5} > \sqrt{4} = 2$, so $\dfrac{1}{\sqrt{5}} < \dfrac{1}{2} < 1$. It follows by the Ratio Test that $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+1)}{5^{n/2}}$ converges. (Absolutely, in fact.) ■

Quiz #21. Monday, 31 March, 2014. *[15 minutes]*

1. Find the radius and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{3^n x^n}{n+1}$. *[5]*

SOLUTION. We will use the Ratio Test to find the radius of convergence R:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{3^{n+1}x^{n+1}}{(n+1)+1}}{\frac{3^n x^n}{n+1}}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}x^{n+1}}{n+2}\cdot\frac{n+1}{3^n x^n}\right| = \lim_{n\to\infty}\left|\frac{3x(n+1)}{n+2}\right| \\
&= 3|x|\lim_{n\to\infty}\frac{n+1}{n+2}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} = 3|x|\lim_{n\to\infty}\frac{1+\frac{1}{n}}{1+\frac{2}{n}} = 3|x|\frac{1+0}{1+0} = 3|x|
\end{aligned}$$

It follows by the Ratio Test that the series will converge absolutely when $3|x| < 1$, *i.e.* when $|x| < \frac{1}{3}$, and diverges when $3|x| > 1$, *i.e.* when $|x| > \frac{1}{3}$. Thus $R = \frac{1}{3}$.

It remains to determine what happens at the endpoints of the interval of convergence, $x = \pm\frac{1}{3}$. When $x = -\frac{1}{3}$, we get $\displaystyle\sum_{n=0}^{\infty} \frac{3^n\left(-\frac{1}{3}\right)^n}{n+1} = \sum_{n=0}^{\infty}\frac{1}{n+1} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$, which is the harmonic series, verified to diverge in class. (You could use the p-Test here: $p = 1 \not< 1$, so it diverges.) When $x = \frac{1}{3}$, we get $\displaystyle\sum_{n=0}^{\infty} \frac{3^n\left(\frac{1}{3}\right)^n}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$, which is the alternating harmonic series, verified to converge (conditionally) in class. (You could use the Alternating Series Test to check it yet again ...) Thus the interval of convergence includes $\frac{1}{3}$, but not $-\frac{1}{3}$, so it is $\left(-\frac{1}{3}, \frac{1}{3}\right]$. ■