Mathematics 1101Y – Calculus I: Functions and calculus of one variable
TRENT UNIVERSITY, 2013–2014
Solutions to the Final Examination

Time: *19:00–22:00, on Tuesday, 15 April, 2014.* *Brought to you by* Стефан Біланюк.

Instructions: Do parts **L**, **M**, and **N**, and, if you wish, part **O**. Show all your work and justify all your answers. *If in doubt about something,* **ask!**

Aids: Any calculator; (all sides of) one aid sheet; one (1) brain (no neuron limit).

Part L. Do all four (4) of **1**–**4**.

1. Compute $\frac{dy}{dx}$ as best you can in any *three* (3) of **a**–**f**. *[15 = 3 × 5 each]*

a. $y = \left(\frac{x+1}{x-1}\right)^2$ **b.** $y = \int_0^x te^{t^2}\,dt$ **c.** $\begin{array}{l} y = -\cos(t) \\ x = \sin(t) \end{array}$

d. $\ln(xy) = 0$ **e.** $y = \sin\left(\sqrt{x}\right)$ **f.** $y = x^\pi e^x$

SOLUTIONS. **a.** Power, Chain, and Quotient Rules:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x+1}{x-1}\right)^2 = 2\left(\frac{x+1}{x-1}\right) \cdot \frac{d}{dx}\left(\frac{x+1}{x-1}\right) \\ &= 2\left(\frac{x+1}{x-1}\right) \cdot \frac{\left[\frac{d}{dx}(x+1)\right](x-1) - (x+1)\left[\frac{d}{dx}(x-1)\right]}{(x-1)^2} \\ &= 2\left(\frac{x+1}{x-1}\right) \cdot \frac{1 \cdot (x-1) - (x+1) \cdot 1}{(x-1)^2} = 2\left(\frac{x+1}{x-1}\right) \cdot \frac{-2}{(x-1)^2} = \frac{-4(x+1)}{(x-1)^3} \quad \blacksquare \end{aligned}$$

b. Using the Fundamental Theorem of Calculus: $\frac{dy}{dx} = \frac{d}{dx}\left(\int_0^x te^{t^2}\,dt\right) = xe^{x^2}$. ■

c. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(-\cos(t)\right)}{\frac{d}{dt}\sin(t)} = \frac{-\left(-\sin(t)\right)}{\cos(t)} = \frac{\sin(t)}{\cos(t)} = \tan(t) = -\frac{x}{y}$. ■

d. $\ln(xy) = 0 \;\Rightarrow\; xy = 1 \;\Rightarrow\; y = \frac{1}{x} \;\Rightarrow\; \frac{dy}{dx} = \frac{d}{dx}x^{-1} = (-1)x^{-2} = -\frac{1}{x^2}$. ■

e. Chain and Power Rules:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\sin\left(\sqrt{x}\right) = \cos\left(\sqrt{x}\right) \cdot \frac{d}{dx}\sqrt{x} = \cos\left(\sqrt{x}\right) \cdot \frac{d}{dx}x^{1/2} \\ &= \cos\left(\sqrt{x}\right) \cdot \frac{1}{2}x^{-1/2} = \frac{\cos\left(\sqrt{x}\right)}{2\sqrt{x}} \quad \blacksquare \end{aligned}$$

f. Product and Power Rules:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^\pi e^x\right) = \left[\frac{d}{dx}x^\pi\right]e^x + x^\pi\left[\frac{d}{dx}e^x\right] = \pi x^{\pi-1}e^x + x^\pi e^x = x^{\pi-1}e^x\left(\pi + x\right) \quad \blacksquare$$

2. Evaluate any *three* (3) of the integrals **a**–**f**. *[15 = 3 × 5 each]*

$$\textbf{a.} \int \frac{e^{\sqrt{t}}}{2\sqrt{t}}\,dt \qquad \textbf{b.} \int_0^{\pi/2} x\cos(x)\,dx \qquad \textbf{c.} \int \sqrt{1-x^2}\,dx$$

$$\textbf{d.} \int_0^{\infty} e^{-y}\,dy \qquad \textbf{e.} \int \frac{x^2+x+1}{x\left(x^2+1\right)}\,dx \qquad \textbf{f.} \int_0^{\pi/4} \tan^2(z)\,dz$$

SOLUTIONS. **a.** We will use the substitution $u = \sqrt{t}$, so $du = \frac{1}{2\sqrt{t}}\,dt$:

$$\int \frac{e^{\sqrt{t}}}{2\sqrt{t}}\,dt = \int e^u\,du = e^u + C = e^{\sqrt{t}} + C \quad \blacksquare$$

b. We will use integration by parts with $u = x$ and $v' = \cos(x)$, so $u' = 1$ and $v = \sin(x)$:

$$\begin{aligned}
\int_0^{\pi/2} x\cos(x)\,dx &= x\sin(x)\Big|_0^{\pi/2} - \int_0^{\pi/2} 1\sin(x)\,dx \\
&= \left[\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) - 0\sin(0)\right] - \left(-\cos(x)\right)\Big|_0^{\pi/2} \\
&= \left[\frac{\pi}{2}\cdot 1 - 0\cdot 0\right] + \left[\cos\left(\frac{\pi}{2}\right) - \cos(0)\right] = \frac{\pi}{2} + [0-1] = \frac{\pi}{2} - 1 \quad \blacksquare
\end{aligned}$$

c. We will use the trigonometric substitution $x = \sin(\theta)$, so $dx = \cos(\theta)\,d\theta$, as well as the trigonometric identity $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$, and eventually the substitution $w = 2\theta$, so $dw = 2\,d\theta$ and $d\theta = \frac{1}{2}\,dw$..

$$\begin{aligned}
\int \sqrt{1-x^2}\,dx &= \int \sqrt{1-\sin^2(\theta)}\,\cos(\theta)\,d\theta = \int \sqrt{\cos^2(\theta)}\,\cos(\theta)\,d\theta \\
&= \int \cos^2(\theta)\,d\theta = \int \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)\,d\theta = \int \left(\frac{1}{2} + \frac{1}{2}\cos(w)\right)\frac{1}{2}\,dw \\
&= \frac{1}{4}\int \left(1+\cos(w)\right)\,dw = \frac{1}{4}\left(w + \sin(w)\right) + C \quad \blacksquare
\end{aligned}$$

d. We will use the substitution $s = -y$, so $ds = -1\,dy$ and $dy = -1\,ds$, and $\begin{smallmatrix} y & 0 & t \\ s & 0 & -t \end{smallmatrix}$.

$$\begin{aligned}
\int_0^{\infty} e^{-y}\,dy &= \lim_{t\to\infty}\int_0^{t} e^{-y}\,dy = \lim_{t\to\infty}\int_0^{-t} e^{s}\,(-1)\,ds = \lim_{t\to\infty} -e^s\Big|_0^{-t} \\
&= \lim_{t\to\infty}\left[-e^{-t} - \left(-e^0\right)\right] = \lim_{t\to\infty}\left[-e^{-t}+1\right] = -0+1 = 1\,,
\end{aligned}$$

since $e^{-t} \to 0$ as $t \to \infty$. $\blacksquare$

e. We will use partial fractions. Since $x^2 + 1 \geq 1 > 0$, the quadratic x^2+1 is irreducible [cannot be factored into linear factors]. It follows that the partial fraction expansion of the integrand has the form

$$\frac{x^2+x+1}{x\left(x^2+1\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A\left(x^2+1\right) + (Bx+C)x}{x\left(x^2+1\right)} = \frac{(A+B)x^2 + Cx + A}{x\left(x^2+1\right)}$$

for some constants A, B, and C. Equating the last numerator with that of the original integrand tells us that $A + B = 1$, $C = 1$, and $A = 1$, from the last and first of which it quickly follows that $B = 0$. One can also achieve the same end with a bit of algebra:

$$\frac{x^2+x+1}{x\left(x^2+1\right)} = \frac{x^2+1}{x\left(x^2+1\right)} + \frac{x}{x\left(x^2+1\right)} = \frac{1}{x} + \frac{1}{x^2+1}$$

Either way,

$$\int \frac{x^2+x+1}{x\left(x^2+1\right)}\,dx = \int \left[\frac{1}{x} + \frac{1}{x^2+1}\right] dx = \ln(x) + \arctan(x) + K\,,$$

where we denote the constant of integration by K, because we already used the symbol C earlier. ■

f. We will use the trigonometric identity $\tan^2(z) = \sec^2(z) - 1$.

$$\begin{aligned}\int_0^{\pi/4} \tan^2(z)\,dz &= \int_0^{\pi/4} \left[\sec^2(z) - 1\right] dz = \left[\tan(z) - z\right]\Big|_0^{\pi/4} \\ &= \left[1 - \frac{\pi}{4}\right] - [0 - 0] = 1 - \frac{\pi}{4} \quad \blacksquare\end{aligned}$$

3. Do any *three* (3) of **a**–**g**. *[15 = 3 × 5 each]*

a. Let $f(x) = x^2 + 1$ and compute $f'(1)$ using the limit definition of the derivative.

b. Find the arc-length of the curve $y = \frac{2}{3}x^{3/2}$, $0 \le x \le 3$.

c. Compute $\lim_{n\to\infty} \frac{n^2}{e^n}$.

d. Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges absolutely or conditionally, or diverges.

e. Sketch the polar curve $r = \sec(\theta)$, $0 \le \theta \le \frac{\pi}{4}$, and find the area of the region between this curve and the origin.

f. Find the number b such that the average value of $y = 1 - x$ on $0 \le x \le b$ is $\frac{1}{2}$.

g. Find the interval of convergence of the power series $\sum_{n=0}^{\infty} 4^{-n}x^n$.

SOLUTIONS. **a.** Here goes:

$$\begin{aligned}f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\left[(1+h)^2 + 1\right] - \left[1^2 + 1\right]}{h} \\ &= \lim_{h\to 0} \frac{\left[1 + 2h + h^2 + 1\right] - 2}{h} = \lim_{h\to 0} \frac{2h + h^2}{h} = \lim_{h\to 0}(2 + h) = 2 + 0 = 2 \quad \blacksquare\end{aligned}$$

b. We plug $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{2}{3}x^{3/2}\right) = \dfrac{2}{3}\cdot\dfrac{3}{2}x^{1/2} = x^{1/2}$ and the limits $0 \le x \le 3$ into the arc-length formula:

$$\text{arc-length} = \int_0^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^3 \sqrt{1+\left(x^{1/2}\right)^2}\,dx = \int_0^3 \sqrt{1+x}\,dx$$

Now we'll substitute $u = x+1$, so $du = dx$ and $\begin{smallmatrix} x & 0 & 3 \\ u & 1 & 4 \end{smallmatrix}$.

$$= \int_1^4 \sqrt{u}\,du = \int_1^4 u^{1/2}\,du = \frac{2}{3}u^{3/2}\Big|_1^4 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3} \quad \blacksquare$$

c. Here goes, using l'Hôpital's Rule twice:

$$\lim_{n\to\infty}\frac{n^2}{e^n} = \lim_{x\to\infty}\frac{x^2 \to \infty}{e^x \to \infty} = \lim_{x\to\infty}\frac{\frac{d}{dx}x^2}{\frac{d}{dx}e^x} = \lim_{x\to\infty}\frac{2x \to \infty}{e^x \to \infty}$$

$$= \lim_{x\to\infty}\frac{\frac{d}{dx}2x}{\frac{d}{dx}e^x} = \lim_{x\to\infty}\frac{2 \to 2}{e^x \to \infty} = 0 \quad \blacksquare$$

d. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ diverges (by the p-Test since $p = \frac{1}{2} \le 1$), $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ does not converge absolutely. On the other hand, the given series does converge by the Alternating Series Test:

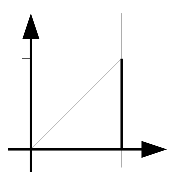
i. Since $\sqrt{n} > 0$ for all $n \ge 1$, $a_n = \dfrac{(-1)^n}{\sqrt{n}}$ alternates sign because $(-1)^n$ does.

ii. Since $\sqrt{x}$ is an increasing function, $\sqrt{n+1} > \sqrt{n}$ for each $n \ge 1$, and so $|a_{n+1}| = \left|\dfrac{(-1)^{n+1}}{\sqrt{n+1}}\right| = \dfrac{1}{\sqrt{n+1}} < \dfrac{1}{\sqrt{n}} = \left|\dfrac{(-1)^n}{\sqrt{n}}\right| = |a_n|$.

iii. $\displaystyle\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\left|\frac{(-1)^n}{\sqrt{n}}\right| = \lim_{n\to\infty}\frac{1 \to 1}{\sqrt{n}\to\infty} = 0.$

Since the series converges, but not absolutely, it converges conditionally. $\blacksquare$

e. Since $x = r\cos(\theta) = \sec(\theta)\cos(\theta) = \frac{\cos(\theta)}{\cos(\theta)} = 1$ for this polar curve, it is a part of the Cartesian line $x = 1$, and because $y = r\sin(\theta) = \sec(\theta)\sin(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$, we have $0 = \tan(0) \le y \le \tan\left(\frac{\pi}{4}\right) = 1$ as $0 \le \theta \le \frac{\pi}{4}$. Here is a sketch:



Since the region between the curve and the origin is a triangle with base and height both equal to 1, it has area $\frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$. $\blacksquare$

f. The average value of $y = 1 - x$ on $0 \le x \le b$ is

$$\frac{1}{b-0}\int_0^b (1-x)\,dx = \frac{1}{b}\left(x - \frac{x^2}{2}\right)\Big|_0^b = \frac{1}{b}\left(b - \frac{b^2}{2}\right) - \frac{1}{b}\left(0 - \frac{0^2}{2}\right) = 1 - \frac{b}{2}.$$

Then $1 - \dfrac{b}{2} = \dfrac{1}{2} \;\Rightarrow\; \dfrac{b}{2} = 1 - \dfrac{1}{2} = \dfrac{1}{2} \;\Rightarrow\; b = 1.$ $\blacksquare$

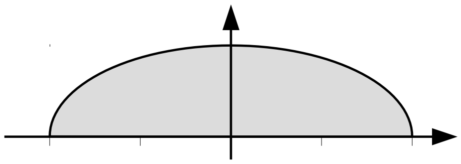
g. $\displaystyle\sum_{n=0}^{\infty} 4^{-n}x^n = \sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$, so this is a geometric series with common ratio $r = \dfrac{x}{4}$. It follows that it converges when $|r| = \left|\frac{x}{4}\right| < 1$, *i.e.* when $|x| < 4$, and diverges when $|r| = \left|\frac{x}{4}\right| \geq 1$, *i.e.* when $|x| \geq 4$, so its interval of convergence is $(-4, 4)$. ■

4. Consider the region below $y = \sqrt{1 - \dfrac{x^2}{4}}$ and above $y = 0$, for $-2 \leq x \leq 2$.

a. Sketch the region and find its area. *[6]*

b. Sketch the solid obtained by revolving this region about the x-axis and find its volume. *[6]*

SOLUTIONS. **a.** The region consists of the upper half of the ellipse $\dfrac{x^2}{4} + y^2 = 1$. Here is a sketch:

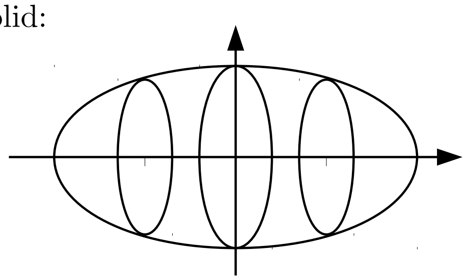


It remains to compute its area. If you remember the area formula for an ellipse, that $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ encloses an area of πab, you can apply it here to get the area, not forgetting that we have only half of an ellipse: $\frac{1}{2}\pi ab = \frac{1}{2}\pi 2 \cdot 1 = \pi$. Otherwise we must integrate:

We will use the trigonometric substitution $x = 2\sin(\theta)$, so $dx = 2\cos(\theta)\,d\theta$, $\sin(\theta) = \dfrac{x}{2}$, $\cos(\theta) = \sqrt{1 - \dfrac{x^2}{4}}$, and $\theta = \arcsin\left(\dfrac{x}{2}\right)$. We will keep the limits, substituting back into x before using them. Along the way, we will also use the trigonometric identity $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$ and the substitution $u = 2\theta$, so $du = 2\,d\theta$.

$$
\begin{aligned}
\text{Area} &= \int_{-2}^{2} \sqrt{1 - \frac{x^2}{4}}\,dx = \int_{x=-2}^{x=2} \sqrt{1 - \frac{(2\sin(\theta))^2}{4}}\,2\cos(\theta)\,d\theta \\
&= 2\int_{x=-2}^{x=2} \sqrt{1 - \sin^2(\theta)}\,\cos(\theta)\,d\theta = 2\int_{x=-2}^{x=2} \cos^2(\theta)\,d\theta \\
&= 2\int_{x=-2}^{x=2} \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)\,d\theta = \int_{x=-2}^{x=2} \left(\frac{1}{2} + \frac{1}{2}\cos(u)\right)\,du \\
&= \left(\frac{1}{2}u + \frac{1}{2}\sin(u)\right)\Bigg|_{x=-2}^{x=2} = \left(\frac{1}{2}2\theta + \frac{1}{2}\sin(2\theta)\right)\Bigg|_{x=-2}^{x=2} \\
&= \left(\arcsin\left(\frac{x}{2}\right) + \frac{1}{2}2\sin(\theta)\cos(\theta)\right)\Bigg|_{x=-2}^{x=2} = \left(\arcsin\left(\frac{x}{2}\right) + x\sqrt{1 - \frac{x^2}{4}}\right)\Bigg|_{x=-2}^{x=2} \\
&= \left(\arcsin\left(\frac{2}{2}\right) + x\sqrt{1 - \frac{2^2}{4}}\right) - \left(\arcsin\left(\frac{-2}{2}\right) + x\sqrt{1 - \frac{(-2)^2}{4}}\right) \\
&= \arcsin(1) + 2 \cdot 0 - \arcsin(-1) - (-2) \cdot 0 = \frac{\pi}{2} - \left(\frac{\pi}{2}\right) = \pi \quad ■
\end{aligned}
$$

b. Here is a sketch of the solid:



We will use the disk/washer method to find the volume of this solid. Since we revovled the region about the x-axis, this means we should use x as the variable of integration. Note that because the lower border of the region is the axis of revolution, each cross-section is a solid disk; the cross-section at x has radius $R = y - 0 = \sqrt{1 - \frac{x^2}{4}}$. Thus:

$$\begin{aligned}
\text{Volume} &= \int_{-2}^{2} \pi R^2 \, dx = \int_{-2}^{2} \pi \left(\sqrt{1 - \frac{x^2}{4}} \right)^2 \, dx = \pi \int_{-2}^{2} \left(1 - \frac{x^2}{4} \right) \, dx \\
&= \pi \left(x - \frac{x^3}{12} \right) \Big|_{-2}^{2} = \pi \left(2 - \frac{2^3}{12} \right) - \pi \left((-2) - \frac{(-2)^3}{12} \right) \\
&= \pi \left(\frac{24}{12} - \frac{8}{12} \right) - \pi \left(-\frac{24}{12} + \frac{8}{12} \right) = \frac{16}{12}\pi - \left(-\frac{16}{12} \right) \pi = \frac{32}{12}\pi = \frac{8}{3}\pi \quad \blacksquare
\end{aligned}$$

Part M. Do any *two* (2) of **5**–**7**. *[28 = 2 × 14 each]*

5. Find the domain and any and all intercepts, vertical and horizontal asymptotes, and maximum, minimum, and inflection points of $f(x) = e^{-x^2}$, and sketch its graph.

Solution. We'll run through the usual checklist and then graph $f(x) = e^{-x^2}$:

i. Domain. Note that both $g(x) = e^x$ and $h(x) = -x^2$ are defined and continuous for all x. It follows that $f(x) = g(h(x)) = e^{-x^2}$ is also defined and continuous for all x. Thus the domain of $f(x)$ is all of $\mathbb{R}$.

ii. Intercepts. Since $g(x) = e^x$ is never 0, $f(x) = e^{-x^2}$ can never equal 0 either, so it has no x-intercepts. For the y-intercept, simply note that $f(0) = e^{-0^2} = e^0 = 1$.

iii. Vertical asymptotes. $f(x) = e^{-x^2}$ is defined and continuous for all x, so it cannot have any vertical asymptotes.

iv. Horizontal asymptotes. We check for horizontal asymptotes:

$$\lim_{x\to\infty} e^{-x^2} = \lim_{x\to\infty} \frac{1}{e^{x^2}} = 0 \quad \text{and} \quad \lim_{x\to-\infty} e^{-x^2} = \lim_{x\to-\infty} \frac{1}{e^{x^2}} = 0\,,$$

since $e^{x^2} \to \infty$ as $x^2 \to \infty$, which happens as $x \to \pm\infty$. Thus $f(x) = e^{-x^2}$ has the horizontal asymptote $y = 0$ in both directions.

v. Maxima and minima. $f'(x) = e^{-x^2} \dfrac{d}{dx}\left(-x^2\right) = -2xe^{-x^2}$, which equals 0 exactly when $x = 0$ because $-2e^{-x^2} \neq 0$ for all x. Note that this is the only critical point. Since $e^{-x^2} > 0$ for all x, $f'(x) = -2xe^{-x^2} > 0$ when $x < 0$ and < 0 when $x > 0$, so $f(x) = e^{-x^2}$ is increasing for $x < 0$ and decreasing for $x > 0$. Thus $x = 0$ is an (absolute!) maximum point of $f(x)$, which has no minimum points. We summarize all this in the usual table:

x	$(-\infty, 0)$	0	$(0, \infty)$
$f'(x)$	$+$	0	1
$f(x)$	$\uparrow$	max	$\downarrow$

vi. Inflection points.

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(-2xe^{-x^2}\right) = -2e^{-x^2} - 2x\frac{d}{dx}\left(-x^2\right) \\ &= -2e^{-x^2} - 2x \cdot \left(-2xe^{-x^2}\right) = \left(4x^2 - 2\right)e^{-x^2}, \end{aligned}$$

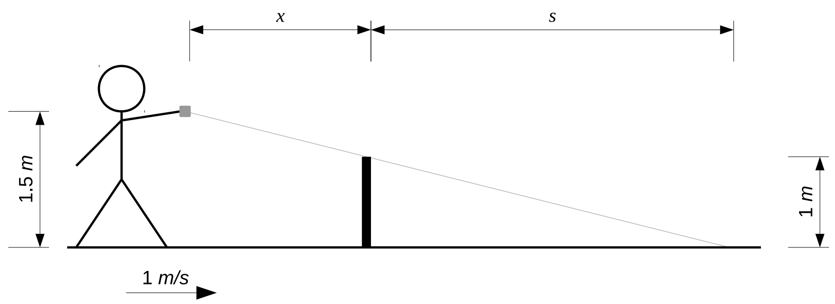
which equals 0 exactly when $4x^2 - 2 = 0$, *i.e.* when $x = \pm\frac{1}{\sqrt{2}}$, because $e^{-x^2} \neq 0$ for all x. Since $e^{-x^2} > 0$ for all x, $f''(x) = \left(4x^2 - 2\right)e^{-x^2} > 0$ exactly when $4x^2 - 2 > 0$, *i.e.* when $|x| > \frac{1}{\sqrt{2}}$, and is < 0 exactly when $4x^2 - 2 < 0$, *i.e.* when $|x| < \frac{1}{\sqrt{2}}$. Thus $f(x) = e^{-x^2}$ is concave up on $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$ and concave down on $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Thus $f(x) = e^{-x^2}$ has two inflection points, at $x = \pm\frac{1}{\sqrt{2}}$. We summarize all this in the usual table:

x	$\left(-\infty, -\frac{1}{\sqrt{2}}\right)$	$-\frac{1}{\sqrt{2}}$	$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{1}{\sqrt{2}}$	$\left(\frac{1}{\sqrt{2}}, \infty\right)$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\smile$	infl. pt.	$\frown$	infl. pt.	$\smile$

vii. Graph. Cheating slightly, this graph was generated using a program called `KAlgebra`.

■

6. Meredith, carrying a lamp 1.5 m above the ground, walks at 1 m/s along level ground directly toward a 1 m tall post at night. How is the length of the shadow cast by the post in the lamplight changing at the instant that the lamp is 2 m from the post?



SOLUTION. Let x be the horizontal distance between the lamp and the post, and let s be the length of the shadow, as in the slightly modified diagram above. We are given that $\frac{dx}{dt} = -1$. By the similarity of the triangles involved, $\frac{x+s}{1.5} = \frac{s}{1}$, so $x + s = 1.5s = \frac{3}{2}s$ and so $x = \frac{1}{2}s$ and $s = 2x$. It follows that $\left.\frac{ds}{dt}\right|_{x=2} = 2\left.\frac{dx}{dt}\right|_{x=2} = 2(-1) = -2\ m/s$. Thus the length of the shadow is decreasing at a rate of 2 m/s at the instant in question. Note that it changes at the same constant rate at every other instant, too. ■

7. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{\sqrt{n\sqrt{n}}}$ converges or diverges.

SOLUTION. A little bit of algebra makes this rather easier to analyze:

$$\begin{aligned}\frac{\sqrt{n+1}-\sqrt{n-1}}{\sqrt{n\sqrt{n}}} &= \frac{\sqrt{n+1}-\sqrt{n-1}}{\sqrt{n\sqrt{n}}} \cdot \frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n-1}} \\ &= \frac{(n+1)-(n-1)}{\sqrt{n\sqrt{n}}\left(\sqrt{n+1}+\sqrt{n-1}\right)} = \frac{2}{\sqrt{n\sqrt{n}}\left(\sqrt{n+1}+\sqrt{n-1}\right)}\end{aligned}$$

Since $\sqrt{n+1}+\sqrt{n-1} \geq \sqrt{n+1} > \sqrt{n} = n^{1/2}$ and $\frac{\sqrt{n+1}-\sqrt{n-1}}{\sqrt{n\sqrt{n}}} > 0$ for $n \geq 1$, and $\sqrt{n\sqrt{n}} = \sqrt{n} \cdot \sqrt{\sqrt{n}}$, it follows that

$$\begin{aligned}0 < \frac{\sqrt{n+1}-\sqrt{n-1}}{\sqrt{n\sqrt{n}}} &= \frac{2}{\sqrt{n\sqrt{n}}\left(\sqrt{n+1}+\sqrt{n-1}\right)} \\ &\leq \frac{2}{\sqrt{n}\cdot\sqrt{\sqrt{n}}\cdot\sqrt{n}} = \frac{2}{n\sqrt{\sqrt{n}}} = \frac{2}{n^{5/4}}\,.\end{aligned}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^{5/4}} = 2\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ converges by the p-Test, because $p = \frac{5}{4} > 1$, it follows by the Basic Comparison Test that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{\sqrt{n\sqrt{n}}}$ converges as well. ■

Part N. Do *one* (1) of **8** or **9**. *[15 = 1 × 15]*

8. Let $f(x) = \sin\left(\frac{x}{2}\right)$.

a. Use Taylor's formula to find the Taylor series at 0 of $f(x)$. *[9]*

b. Find the radius and interval of convergence of this Taylor series. *[6]*

c. *[Bonus!]* Verify that the Taylor series at 0 of $f(x)$ actually converges to $f(x)$. *[1]*

SOLUTIONS. **a.** Note first that $\frac{d}{dx}\sin\left(\frac{x}{2}\right) = \cos\left(\frac{x}{2}\right)\cdot\frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}\cos\left(\frac{x}{2}\right)$ and $\frac{d}{dx}\cos\left(\frac{x}{2}\right) = -\sin\left(\frac{x}{2}\right)\cdot\frac{d}{dx}\left(\frac{x}{2}\right) = -\frac{1}{2}\sin\left(\frac{x}{2}\right)$. Along with the facts that $\sin(0) = 0$ and $\cos(0) = 1$, we can use this information to build the following table:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin\left(\frac{x}{2}\right)$	0
1	$\frac{1}{2}\cos\left(\frac{x}{2}\right)$	$\frac{1}{2}$
2	$-\frac{1}{4}\sin\left(\frac{x}{2}\right)$	0
3	$-\frac{1}{8}\cos\left(\frac{x}{2}\right)$	$-\frac{1}{8}$
4	$\frac{1}{16}\sin\left(\frac{x}{2}\right)$	0
5	$\frac{1}{32}\cos\left(\frac{x}{2}\right)$	$\frac{1}{32}$
$\vdots$	$\vdots$	$\vdots$
$2k$	$\frac{(-1)^k}{2^{2k}}\sin\left(\frac{x}{2}\right)$	0
$2k+1$	$\frac{(-1)^k}{2^{2k+1}}\cos\left(\frac{x}{2}\right)$	$\frac{(-1)^k}{2^{2k+1}}$
$\vdots$	$\vdots$	$\vdots$

The pattern in the last column tells us that every even power in the Taylor series at 0 of $f(x) = \sin\left(\frac{x}{2}\right)$ will have a coefficient of 0, so we only need to account for the odd terms. Thus the Taylor series at 0 of $f(x) = \sin\left(\frac{x}{2}\right)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{k=0}^{\infty} \frac{\frac{(-1)^k}{2^{2k+1}}}{(2k+1)!}x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}(2k+1)!}. \quad \blacksquare$$

b. We will use the Ratio Test to find the radius of convergence of the Taylor series obtained in the solution to **a** above.

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\frac{(-1)^{k+1}x^{2(k+1)+1}}{2^{2(k+1)+1}(2(k+1)+1)!}}{\frac{(-1)^k x^{2k+1}}{2^{2k+1}(2k+1)!}}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}x^{2k+3}}{2^{2k+3}(2k+3)!}\cdot\frac{2^{2k+1}(2k+1)!}{(-1)^k x^{2k+1}}\right|$$

$$= \lim_{k\to\infty}\left|\frac{(-1)x^2}{2^2(2k+3)(2k+1)}\right| = \frac{x^2}{4}\lim_{k\to\infty}\frac{1}{(2k+3)(2k+1)}\begin{matrix}\to 1\\ \to\infty\end{matrix} = \frac{x^2}{4}\cdot 0 = 0$$

Since the limit of 0 does not depend on x and $0 < 1$, it follows by the Ratio Test that the series converges for all $x \in \mathbb{R}$. Thus the radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \infty)$. $\blacksquare$

c. Recall that if $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$ is the Taylor polynomial of degree n, also known as the n'th partial sum of the Taylor series at 0 of $f(x)$, then the nth remainder term is $R_n(x) = f(x) - T_n(x)$. Lagrange showed that $R_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!} x^{n+1}$ for some t between 0 and x.

We will use the Lagrange form of the remainder term to verify that the Taylor series obtained in the solution to **a** actually converges to $f(x) = \sin\left(\frac{x}{2}\right)$. From the second column of the table in the solution to **a**, $f^{(n+1)}(t)$ must be $\pm\frac{1}{2^{n+1}} \sin\left(\frac{t}{2}\right)$ or $\pm\frac{1}{2^{n+1}} \cos\left(\frac{t}{2}\right)$. Either way, because both $\left|\sin\left(\frac{t}{2}\right)\right| \leq 1$ and $\left|\cos\left(\frac{t}{2}\right)\right| \leq 1$ for all t, we get that $\left|f^{(n+1)}(t)\right| \leq \frac{1}{2^{n+1}}$. It follows that

$$0 \leq |R_n(x)| \leq \left| \frac{f^{(n+1)}(t)}{(n+1)!} x^{n+1} \right| \leq \frac{|x|^{n+1}}{2^{n+1}(n+1)!},$$

and, since $\lim_{n\to\infty} \frac{|x|^{n+1}}{2^{n+1}(n+1)!} = 0$ (factorials beat exponential functions as $n \to \infty$), it follows by the Squeeze Theorem that $\lim_{n\to\infty} |R_n(x)| = 0$. As this did not depend on the value of x and $R_n(x) = f(x) - T_n(x)$, it follows that $T_n(x) \to f(x)$ as $n \to \infty$ for all x, *i.e.* the series converges to $f(x)$ for all x. ■

9. Suppose $f(x)$ has $\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$ as its Taylor series at 0.

a. Find the radius and interval of convergence of this Taylor series. *[6]*

b. Use Taylor's formula to determine $f^{(n)}(0)$ for $n \geq 0$. *[9]*

c. *[Bonus!]* Find a formula, other than the series, for $f(x)$. *[1]*

SOLUTIONS. **a.** We will use the Ratio Test to find the radius of convergence of the given Taylor series at 0.

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{\frac{x^{2(k+1)}}{(2(k+1))!}}{\frac{x^{2k}}{(2k)!}} \right| = \lim_{k\to\infty} \left| \frac{x^{2k+2}}{(2k+2)!} \cdot \frac{(2k)!}{x^{2k}} \right| = \lim_{k\to\infty} \left| \frac{x^2}{(2k+2)(2k+1)} \right|$$

$$= x^2 \lim_{k\to\infty} \frac{1}{(2k+2)(2k+1)} \begin{matrix} \to 1 \\ \to \infty \end{matrix} = x^2 \cdot 0 = 0$$

Since the limit of 0 does not depend on x and $0 < 1$, it follows by the Ratio Test that the series converges for all $x \in \mathbb{R}$. Thus the radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \infty)$. ■

b. Taylor's formula tells us that the coefficient of x^n in the Taylor series at 0 is $\frac{f^{(n)}(0)}{n!}$. The given series has only even powers of x appear, so we must have $f^{(n)}(0) = 0$ for all odd n. On the other hand, if $n = 2k$ is an even integer, the coefficient of $x^n = x^{2k}$ is $\frac{f^{(2k)}(0)}{(2k)!} = \frac{1}{(2k)!}$, so $f^{(n)}(0) = f^{(2k)}(0) = 1$. Thus $f^{(n)}(0) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$. ■

c. Observe that the series given for $f(x)$ is just like the Taylor series at 0 of $\cos(x)$, namely $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$, except for the alternating sign that the $(-1)^k$ provides the series for $\cos(x)$. This should lead one to suspect that $f(x)$ is a modification of $\cos(x)$, with the modification cancelling out the alternating sign. In turn, this suggests that we should plug some modification of x into $\cos(x)$, instead of just plain x, that will make the sign of x^{2k} alternate just like $(-1)^k$, which would make the $2k$th term be $\dfrac{(-1)^k(-1)^k x^{2k}}{(2k)!} = \dfrac{(-1)^{2k} x^{2k}}{(2k)!} = \dfrac{x^{2k}}{(2k)!}$, as desired.

The simplest kind of modification of x that ought to do this would be to multiply x by some number α with the property that $\alpha^{2k} = \left(\alpha^2\right)^k = (-1)^k$, which would mean that $\alpha^2 = -1$. There is no such real number, of course, but the complex numbers have such a number, namely $i = \sqrt{-1}$. Checking, we plug ix in for x into the Taylor series for $\cos(x)$ at 0 to get the Taylor series for $\cos(ix)$ at 0:

$$\sum_{k=0}^{\infty} \frac{(-1)^k (ix)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k i^{2k} x^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k (-1)^k x^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!},$$

which is the Taylor series of $f(x)$ at 0, as desired. Since the Taylor series for $\cos(x)$ at 0 converges to $\cos(x)$, as noted in class, it follows that $f(x) = \cos(ix)$. Note the use of the complex number i in the definition of a real-valued function of a real variable!

An alternative approach is to notice that the given series consists of all of the even powers in the Taylor series of e^x at 0. How then can we get rid of the odd powers? Since $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ [we checked in class that the Taylor series at 0 of e^x does converge to e^x for all x] we get that $e^{-x} = \displaystyle\sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$. Then

$$\begin{aligned}
\cosh(x) &= \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(\left[\sum_{n=0}^{\infty} \frac{x^n}{n!}\right] + \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\right]\right) \\
&= \frac{1}{2}\left(\left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right] + \left[1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right]\right) \\
&= \frac{1}{2}\left(2 + 2\frac{x^2}{2} + 2\frac{x^4}{24} + \cdots\right) = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}.
\end{aligned}$$

Thus $f(x) = \cosh(x)$.

Note that by combining the two arguments we get that $\cosh(x) = \cos(ix)$, just in case you wanted to know how they are truly related. ∎

[Total = 100]

Part O. Bonus problems! If you feel like it and have the time, do one or both of these.

○. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$. Assuming this is so [which it is], what is the series $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = 1 + \frac{1}{9} + \frac{1}{25} + \cdots$ equal to? *[1]*

SOLUTION. A little algebra goes along way here. Note that since the series are compsed entirely of positive terms and converge, they converge absolutely, and so can be rearranged at will without changing the sum. Since

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2} &= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \left(1 + \frac{1}{9} + \frac{1}{25} + \cdots\right) + \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \cdots\right) \\ &= \left(1 + \frac{1}{9} + \frac{1}{25} + \cdots\right) + \frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{9} + \cdots\right) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2},\end{aligned}$$

it follows that $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{3}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{8}$. ■

⊙. Write a haiku touching on calculus or mathematics in general. *[1]*

What is a haiku?

seventeen in three:
five and seven and five of
syllables in lines

SOLUTION. None given! ☺

HAVE SOME FUN THIS SUMMER,
AND DROP BY NEXT YEAR TO TELL ME ABOUT IT!