**Mathematics 1101Y – Calculus I: Functions and calculus of one variable**
TRENT UNIVERSITY, 2012–2013

**Solutions to the Quizzes**

**Quiz #0.** Friday, 14 September, 2012. *[15 minutes]*

Mathematicians in medieval India traditionally wrote up much of their work in verse. The problem below was posed by Bhaskara (*c.* 1114-1185 A.D.) in a book dedicated to his daughter Lilavati:

The square root of half the number of bees in a swarm
Has flown out upon a jasmine bush;
Eight ninths of the swarm has remained behind;
And a female bee flies about a male who is buzzing inside a lotus flower;
In the night, allured by the flower's sweet odour, he went inside it
And now he is trapped!
Tell me, most enchanting lady, the number of bees.

**1.** Restate the problem given above as an equation. *[5]*

**Bonus.** Solve the equation you obtained in your solution to **1**. *[1]*

NOTE: For those interested in the history of mathematics, Bhaskara developed a number of techniques that anticipated portions of both differential and integral calculus. The translation given above of Bhaskara's problem is taken from *The Heritage of Thales*, by W.S. Anglin & J. Lambeck, Springer Verlag, New York, 1995, ISBN 0-387-94544-X, p. 113. Just in case you look up Bhaskara, there was also an earlier (*c.* 600-680 A.D.) notable Indian mathematician with the same name. They sometimes end up being numbered to distinguish them.

SOLUTION TO **1**. If $x$ is the total number of bees in the swarm, the problem tells us that $\sqrt{\frac{x}{2}}$ of them have flown to the jasmine bush, $\frac{8}{9}x$ have remained behind, and 2 more are in or around the lotus flower. Thus

$$x = \sqrt{\frac{x}{2}} + \frac{8}{9}x + 2$$

is the equation given in the problem. ■

SOLUTION $i$ TO THE **Bonus** QUESTION. Here goes! We first rearrange the equation to help isolate and get rid of the square root, and then eliminate all the fractions among the coefficients:

$$x = \sqrt{\frac{x}{2}} + \frac{8}{9}x + 2 \implies \sqrt{\frac{x}{2}} = \frac{1}{9}x - 2 \implies \frac{x}{2} = \left(\frac{1}{9}x - 2\right)^2 = \frac{1}{81}x^2 - \frac{4}{9}x + 4$$
$$\implies \frac{1}{81}x^2 - \frac{17}{18}x + 4 = 0 \implies 2x^2 - 153x + 648 = 0$$

... and then solve the resulting quadratic equation using the quadratic formula:

$$\begin{aligned} x &= \frac{-(-153) \pm \sqrt{(-153)^2 - 4 \cdot 2 \cdot 648}}{2 \cdot 2} = \frac{153 \pm \sqrt{23409 - 5184}}{4} \\ &= \frac{153 \pm \sqrt{18225}}{4} = \frac{153 \pm 135}{4} = \begin{matrix} 288/4 \\ \text{or } 18/4 \end{matrix} = \begin{matrix} 72 \\ \text{or } 9/2 \end{matrix} \end{aligned}$$

Thus $x$, the number of bees in the swarm, is either 72 or $\frac{9}{2}$. Bhaskara didn't allow for fractional bees ... ■

NOTE. Bhaskara's problem has something to do – kind of, sort of – with a *Monty Python* sketch: the 1972 album *Monty Python's Previous Record* includes the song *Eric the Half-a-bee*, where it concludes a variant of their classic *Fish License* sketch. You can find the song on YouTube at:

`http://www.youtube.com/watch?v=MlrsqGal64w`

This song includes the immortal line, *"For half a bee, philosophically, must ipso facto half not be."* Some of the group had, it seems, studied philosophy ...

SOLUTION *ii* TO THE **Bonus** QUESTION. Once more, with `Maple`! The basic tool used here is `Maple`'s `solve` command, which has many options and variations.

Plugging the original equation into `Maple` gives:

```
> solve(x=sqrt(x/2)+8*x/9+2,x);
```

$$72$$

That is, $x = 72$ is the solution to the equation described by Bhaskara.

On the other hand, plugging the equivalent quadratic equation into `Maple` gives:

```
> solve(2*x^2-153*x+648-0,x);
```

$$\frac{9}{2}, 72$$

That is, $x = \frac{9}{2}$ and $x = 72$ are the two solutions of the quadratic equation we obtained and then solved in **2**. Why doesn't `Maple` give $\frac{9}{2}$ as a solution of the original equation? Beats me! ■

**Quiz #1.** Friday, 21 September, 2012. *[10 minutes]*

Consider the parabola $y = x^2 + 4x - 5$.

1. Find the $x$-intercepts of the parabola. *[2]*
2. Find the coordinates of the vertex of the parabola. *[2]*
3. Sketch the graph of the parabola. *[1]*

SOLUTION TO 1. The $x$-intercepts occur when $y = 0$, so we need to find the roots of $x^2 + 4x - 5$.

One way to do this is to factor $x^2 + 4x - 5$; however you do so, you should get $x^2 + 4x - 5 = (x+5)(x-1)$. It then follows that the roots are $x = -5$ (so $x + 5 = 0$) and $x = 1$ (so $x - 1 = 0$).

The other major way to get the job done is to use the quadratic formula: $x^2+4x-5 = 0$ when

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot (-5)}}{2 \cdot 1} = \frac{-4 \pm \sqrt{16+20}}{2} \\ &= \frac{-4 \pm \sqrt{36}}{2} = \frac{-4 \pm 6}{2} = -2 \pm 3 = 1 \text{ or } -5\,. \qquad \blacksquare \end{aligned}$$

SOLUTION TO 2. One way to locate the vertex in this case is to observe that it must occur halfway between the roots, at $x = -2$, and get the $y$-coordinate by plugging $x = -2$ into the equation of the parabola: $y = (-2)^2 + 4(-2) - 5 = 4 - 8 - 5 = -9$.

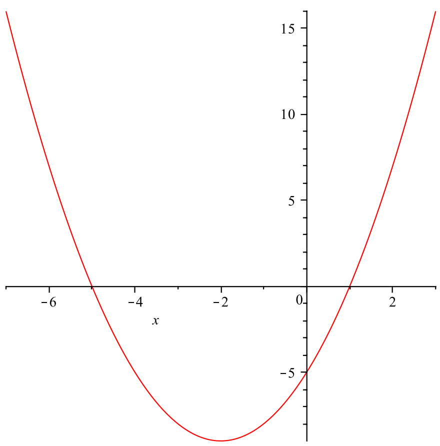
Another way is to complete the square in the equation of the parabola:

$$x^2 + 4x - 5 = \left[x^2 + 4x\right] - 5 = \left[\left(x + \frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2\right] - 5 = (x+2)^2 - 9 = (x - (-2))^2 - 9$$

It follows that the vertex of the parabola is at $(-2, -9)$. ■

SOLUTION TO 3. I cheated and asked a `Maple` to do the job for me:

```
> plot(x^2+4x-5,x=-7..3)
```




■

**Quiz #2.** Friday, 28 September, 2012. *[10 minutes]*

1. Find the inverse function, as best you can, of $f(x) = \dfrac{2e^x}{e^x + 1}$. *[5]*

SOLUTION. As usual, we'll set $x = f(y)$ and try to solve for $y$:

$$\begin{aligned} x = f(y) = \frac{2e^y}{e^y + 1} \quad &\implies \quad x\left(e^y + 1\right) = 2e^y \quad \implies \quad xe^y + x - 2e^y = 0 \\ &\implies \quad (x-2)e^y + x = 0 \quad \implies \quad (x-2)e^y = -x \\ &\implies \quad e^y = \frac{-x}{x-2} = \frac{x}{2-x} \\ &\implies \quad y = \ln\left(\frac{x}{2-x}\right) = \ln(x) - \ln(2-x) \end{aligned}$$

Thus $f^{-x}(x) = \ln\left(\dfrac{x}{2-x}\right) = \ln(x) - \ln(2-x)$.

That's enough for a complete solution, but for those who worry about domains, note that $\ln(x) - \ln(2-x)$ makes sense whenever both $x > 0$ and $2 - x > 0$, *i.e.* $0 < x < 2$. By contrast, $\ln\left(\dfrac{x}{2-x}\right)$ makes sense when $\frac{x}{2-x} > 0$, which happens when either both $x > 0$ and $2 - x > 0$, *i.e.* $0 < x < 2$, or when both $x < 0$ and $2 - x < 0$, *i.e.* $2 < x < 0$, which last doesn't happen for all that many $x$'s ... ■

**Quiz #3.** Friday, 5 October, 2012. *[10 minutes]*

1. Compute $\displaystyle\lim_{x \to 1} \frac{x^4 - 1}{x - 1}$. *[5]*

SOLUTION. Note that both the numerator and denominator approach 0 as $x$ approaches 1. The key to this problem is the fact that the numerator factors as follows:

$$x^4 - 1 = \left(x^2\right)^2 - 1^2 = \left(x^2 - 1\right)\left(x^2 + 1\right) = (x-1)(x+1)\left(x^2+1\right)$$

Thus

$$\begin{aligned} \lim_{x \to 1} \frac{x^4 - 1}{x - 1} &= \lim_{x \to 1} \frac{(x-1)(x+1)\left(x^2+1\right)}{x-1} \\ &= \lim_{x \to 1} (x+1)\left(x^2+1\right) = (1+1)\left(1^2+1\right) = 2 \cdot 2 = 4 \qquad ■ \end{aligned}$$

ALTERNATIVE METHODS. If one didn't spot the repeated difference of squares gimmick used to factor the numerator above, one could divide $x - 1$ into $x^4 - 1$ using long division instead. One could also take advantage of the fact that both numerator and denominator approach 0 and use l'Hôpital's Rule. (Of course, this does require a little knowledge of differentiation.)

**Quiz #4.** Friday, 12 October, 2012. *[10 minutes]*

1. Compute $\lim_{x\to\infty} \frac{x^2}{(3x+1)^2}$. *[5]*

SOLUTION. Here goes:

$$\begin{aligned}\lim_{x\to\infty} \frac{x^2}{(3x+1)^2} &= \lim_{x\to\infty} \frac{x^2}{9x^2+6x+1} = \lim_{x\to\infty} \frac{x^2}{9x^2+6x+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\ &= \lim_{x\to\infty} \frac{1}{9+\frac{6}{x}+\frac{1}{x^2}} = \frac{1}{9+0+0} = \frac{1}{9},\end{aligned}$$

since $\frac{6}{x} \to 0$ and $\frac{1}{x^2} \to 0$ as $x \to \infty$. ■

**Quiz #5.** Friday, 2 November, 2012. *[10 minutes]*

1. Compute $\frac{d}{dx}\left(\frac{\cos^2(x)}{e^x}\right)$. *[5]*

SOLUTION. We will use the Quotient, Power, and Chain Rules:

$$\begin{aligned}\frac{d}{dx}\left(\frac{\cos^2(x)}{e^x}\right) &= \frac{\left(\frac{d}{dx}\cos^2(x)\right)\cdot e^x - \cos^2(x)\cdot\left(\frac{d}{dx}e^x\right)}{\left(e^x\right)^2} \\ &= \frac{\left(2\cos(x)\cdot\left[\frac{d}{dx}\cos(x)\right]\right)\cdot e^x - \cos^2(x)\cdot e^x}{\left(e^x\right)^2} \\ &= \frac{\left(2\cos(x)\cdot[-\sin(x)]\right)\cdot e^x - \cos^2(x)\cdot e^x}{\left(e^x\right)^2} \\ &= \frac{-2e^x\cos(x)\sin(x) - e^x\cos^2(x)}{\left(e^x\right)^2} \\ &= \frac{-2\cos(x)\sin(x) - \cos^2(x)}{e^x} \\ &= -e^{-x}\cos(x)\left[2\sin(x)+\cos(x)\right]\end{aligned}$$

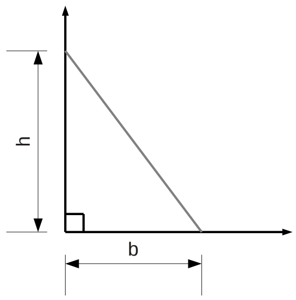
There are, of course, many other ways to get the job done – *e.g.* rewrite $\frac{\cos^2(x)}{e^x}$ as $e^{-x}\cos^2(x)$ first and then use the Product Rule instead of the Quotient Rule – as well as many ways in which the final answer may be written. ■

**Quiz #6.** Friday, 9 November, 2012. *[15 minutes]*

1. Elvis and Solovey start running from the origin at the same time. Elvis runs up the $y$-axis at 8 $m/s$ and Solovey runs right along the $x$-axis at 6 $m/s$. How is the area of the triangle whose corners are the origin, Elvis, and Solovey, changing 2 $s$ after the start? *[5]*

   *Note:* Just in case, the area of triangle with base $b$ and height $h$ is $\frac{1}{2}bh$.

SOLUTION. Suppose that we think of the side of the triangle along the $x$-axis as the base and the side along the $y$-axis as the height. (The axes meet at right angles!)




Then we have $\frac{dh}{dt} = 8\ m/s$ and $\frac{db}{dt} = 6\ m/s$, and it should be pretty obvious that after 2 $s$ we must have $h = 8 \cdot 2 = 16\ m$ and $b = 6 \cdot 2 = 12\ m$. It follows, with a bit of help from the Product Rule, that

$$\begin{aligned} \left.\frac{dA}{dt}\right|_{t=2} &= \frac{d}{dt}\left(\frac{1}{2}bh\right) = \frac{1}{2}\left(\left[\frac{db}{dt}\right] \cdot h + b \cdot \left[\frac{dh}{dt}\right]\right) \\ &= \frac{1}{2}(6 \cdot 16 + 12 \cdot 8) = \frac{1}{2}(96 + 96) = 96\ m^2/s\,. \end{aligned}$$ ■

**Quiz #7.** Friday, 16 November, 2012. *[15 minutes]*

1. Find the domain and all the intercepts, vertical and horizontal asymptotes, maximum and minimum points, and points of inflection of $f(x) = xe^{-x}$ and sketch its graph using this information. *[5]*

SOLUTION. We run through the usual checklist:

*i. Domain.* $f(x) = xe{-}x$ is obviously defined for all $x$ and is continuous everywhere it is defined because it is the composition and product of everywhere continuous functions.

*ii. Intercepts.* $f(0) = 0e^{-0} = 0 \cdot 1 = 0$ so $f(x)$ has a $y$-intercept of 0.
$f(x) = xe^{-x} = 0$ exactly when $x = 0$ (since $e^{-x} > 0$ for all $x$), so the $y$-intercept is also the only $x$-intercept in this case.

*iii. Vertical asymptotes.* Since $f(x)$ is defined and continuous for all $x$, it does not have any vertical asymptotes.

*iv. Horizontal asymptotes.* We check the relevant limits in both directions:

$$\lim_{x\to-\infty} xe^{-x} = \lim_{x\to-\infty} \frac{x \quad \to -\infty}{e^x \quad \to 0^+} = -\infty$$

$$\lim_{x\to+\infty} xe^{-x} = \lim_{x\to+\infty} \frac{x \quad \to +\infty}{e^x \quad \to +\infty} = \lim_{x\to+\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^x} \quad \text{(using l'Hôpital's Rule)}$$

$$= \lim_{x\to+\infty} \frac{1 \quad \to 1}{e^x \quad \to +\infty} = 0^+$$

It follows that we have a horizontal asymptote of $y = 0$ in the positive direction only.

*v. Increase/decrease, etc.* First, a little of the Product and Chain Rules:

$$f'(x) = \frac{d}{dx}\left(xe^{-x}\right) = \left(\frac{d}{dx}x\right)e^{-x} + x\left(\frac{d}{dx}e^{-x}\right)$$
$$= 1e^{-x} + xe^{-x}\cdot\frac{d}{dx}(-x) = e^{-x} + xe^{-x}(-1) = (1-x)e^{-x}$$

Since $e{-}x > 0$ for all $x$, $f'(x) = 0$ precisely when $1 - x = 0$, *i.e.* when $x = 1$. When $x > 1$, $1 - x < 0$, and hence $f'(x) < 0$ and so $f(x)$ is decreasing, and when $x < 1$, $1 - x > 0$, and hence $f'(x) > 0$ and so $f(x)$ is increasing. It follows that the critical point at $x = 1$ gives a (local) maximum of $f(x)$.
The usual table summarizing all this:

| $x$ | $(-\infty, 1)$ | $1$ | $(1, \infty)$ |
|---|---|---|---|
| $f'(x)$ | $+$ | $0$ | $-$ |
| $f(x)$ | $\uparrow$ | max | $\downarrow$ |

*vi. Curvature, etc.* First, a little more of the Product and Chain Rules:

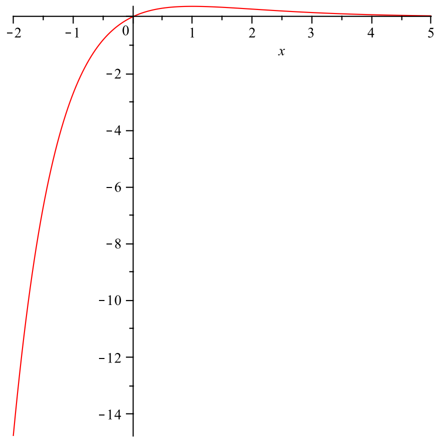
$$f''(x) = \frac{d}{dx}\left((1-x)e^{-x}\right) = \left(\frac{d}{dx}(1-x)\right)e^{-x} + (1-x)\left(\frac{d}{dx}e^{-x}\right)$$
$$= (-1)e^{-x} + (1-x)e^{-x}\cdot\frac{d}{dx}(-x) = -e^{-x} + (1-x)e^{-x}(-1) = (x-2)e^{-x}$$

Since $e{-}x > 0$ for all $x$, $f''(x) = 0$ precisely when $x - 2 = 0$, *i.e.* when $x = 2$. When $x > 2$, $x - 2 > 0$, and hence $f''(x) > 0$ and so $f(x)$ is concave up, and when $x < 2$, $x - 2 < 0$, and hence $f'(x) < 0$ and so $f(x)$ is concave down. It follows that the point at $x = 2$ is a an inflection point of $f(x)$.
The usual table summarizing all this:

| $x$ | $(-\infty, 12)$ | $2$ | $(2, \infty)$ |
|---|---|---|---|
| $f''(x)$ | $-$ | $0$ | $+$ |
| $f(x)$ | $\frown$ | max | $\smile$ |

*vi. The graph.* I cheated and asked a `Maple` to do the job for me:

```
> plot(x*exp(-x),x=-2..5)
```




■

**Quiz #8.** Friday, 23 November, 2012. *[10 minutes]*

1. Find the maximum and minimum values of $f(x) = x^3 - 2x^2 + x$ on the interval $[-1, 1]$. *[5]*

SOLUTION. Note that $f(x)$ is a polynomial, hence defined (and continuous and differentiable) everywhere. It follows that we need not worry about vertical asymptotes or other discontinuities, or points where the derivative is not defined, when solving the problem.

To find the critical points of $f(x)$, we need the derivative:

$$f'(x) = \frac{d}{dx}\left(x^3 - 2x^2 + x\right) = 3x^2 - 4x + 1$$

Using the quadratic formula, $f'(x) = 3x^2 - 4x + 1 = 0$ when

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 3 \cdot 1}}{2 \cdot 3} = \frac{4 \pm \sqrt{4}}{6} = \frac{4 \pm 2}{6} = \frac{2 \pm 1}{3} = \frac{1}{3} \quad \text{or} \quad 1\,.$$

Both of the critical points are in the given interval (though one is also an endpoint), so we compare the values of $f(x)$ at these points and at the endpoints of the interval:

| $x$ | $f(x)$ |
|---|---|
| $-1$ | $-4$ |
| $\frac{1}{3}$ | $\frac{4}{27}$ |
| $1$ | $0$ |

Thus the maximum value of $f(x)$ on $[-1, 1]$ is $\frac{4}{27}$, which is achieved at $x = \frac{1}{3}$, and the minimum value is $-4$, which is achieved at $x = -1$. ■

**Quiz #9.** Friday, 30 November, 2012. *[10 minutes]*

Do *one* (1) of questions 1 or 2 below.

1. Find the maximum area of a right triangle whose hypotenuse has length $\sqrt{8}\,m$. *[5]*

2. Compute $\displaystyle\int_0^{\pi/4} \cos(2x)\,dx$. *[5]*

SOLUTION TO 1. Suppose the base of the right triangle has length $x$ and the height has length $y$. Then $x^2 + y^2 = \left(\sqrt{8}\right)^2 = 8$, so $y = \sqrt{8 - x^2}$, and the area of the triangle is $A = \frac{1}{2}xy = \frac{1}{2}x\sqrt{8 - x^2}$. It should be pretty obvious that the least $x$ can be is 0, and that the most it can be (when $y = 0$) is $\sqrt{8}$. We therefore need to maximize $A = \frac{1}{2}x\sqrt{8 - x^2}$ on the interval $\left[0, \sqrt{8}\right]$. Any critical points in the interval?

$$\begin{aligned}
\frac{dA}{dx} &= \frac{d}{dx}\left[\frac{1}{2}x\sqrt{8-x^2}\right] = \frac{1}{2}\cdot\left[\frac{d}{dx}x\right]\cdot\sqrt{8-x^2} + \frac{1}{2}\cdot x\cdot\left[\frac{d}{dx}\sqrt{8-x^2}\right] \\
&= \frac{1}{2}\cdot 1\cdot\sqrt{8-x^2} + \frac{1}{2}\cdot x\cdot\frac{1}{2\sqrt{8-x^2}}\cdot\frac{d}{dx}\left(8-x^2\right) \\
&= \frac{1}{2}\sqrt{8-x^2} + \frac{1}{2}x\cdot\frac{1}{2\sqrt{8-x^2}}\cdot(-2x) = \frac{\sqrt{8-x^2}}{2} - \frac{x^2}{2\sqrt{8-x^2}} \\
&= \frac{\sqrt{8-x^2}\cdot\sqrt{8-x^2}}{2\sqrt{8-x^2}} - \frac{x^2}{2\sqrt{8-x^2}} = \frac{8-x^2}{2\sqrt{8-x^2}} - \frac{x^2}{2\sqrt{8-x^2}} = \frac{8-2x^2}{2\sqrt{8-x^2}}
\end{aligned}$$

which $= 0$ exactly when $8 - 2x^2 = 0$, *i.e.* when $x^2 = 4$ or $x = \pm 2$. Note that only $x = 2$ is in the given interval. We compare the values of $A$ at the endpoints of the interval to its value at the critical point $x = 2$:

| $x$ | $A$ |
|---|---|
| 0 | 0 |
| 2 | 2 |
| $\sqrt{8}$ | 0 |

It follows that the maximum area of a right triangle whose hypotenuse has length $\sqrt{8}\,m$ is $2\,m^2$. ■

SOLUTION TO 2. We will use the substitution $u = 2x$, so $du = 2dx$ and hence $dx = \frac{1}{2}du$, and change the limits of integration as we go along: $\begin{matrix} x & 0 & \pi/4 \\ u & 0 & \pi/2 \end{matrix}$ Here goes:

$$\begin{aligned}
\int_0^{\pi/4} \cos(2x)\,dx &= \int_0^{\pi/2} \cos(u)\,\frac{1}{2}du = \frac{1}{2}\int_0^{\pi/2}\cos(u)\,du = \frac{1}{2}\sin(u)\Big|_0^{\pi/2} \\
&= \frac{1}{2}\sin(\pi/2) - \frac{1}{2}\sin(0) = \frac{1}{2}\cdot 1 - \frac{1}{2}\cdot 0 = \frac{1}{2} \qquad ■
\end{aligned}$$

**Quiz #10.** Wednesday, 5 December, 2012. *[10 minutes]*

1. Compute $\displaystyle\int 26x^{12}\ln(x)\,dx$. *[5]*

SOLUTION. We will use integration by parts with $u = \ln(x)$ and $v' = 26x^{12} = 2 \cdot 13x^{12}$, so $u' = \dfrac{1}{x}$ and $v = 2x^{13}$.

$$\begin{aligned}\int x^{12}\ln(x)\,dx &= \int uv'\,dx = uv - \int u'v\,dx = \ln(x)\cdot 2x^{13} - \int \frac{1}{x}\cdot 2x^{13}\,dx \\ &= 2x^{13}\ln(x) - \int 2x^{12}\,dx = 2x^{13}\ln(x) - 2\frac{x^{13}}{13} + C \\ &= 2x^{13}\ln(x) - \frac{2}{13}x^{13} + C \qquad \blacksquare\end{aligned}$$

**Quiz #11.** Friday, 11 January, 2013. *[10 minutes]*

1. Find the area between the curves $y = x\ln(x)$ and $y = x$ for $1 \le x \le e$. *[5]*

SOLUTION. Since $ab = a$ either when $a = 0$ or when $b = 1$, $x\ln(x) = x$ should happen when either $x = 0$ or when $\ln(x) = 1$. $x = 0$ is not in the interval under consideration (nor in the domain of $\ln(x)$ ... ), but $\ln(x) = 1$ exactly when $x = e$. Since at $x = 1$, $x = 1 > 0 = 1 \cdot 0 = x\ln(x)$, it follows that for all $x$ with $1 \le x \le e$, we have $x \ge x\ln(x)$. Thus the area between the two curves is :

$$\text{Area} = \int_1^e [x - x\ln(x)]\,dx = \int_1^e x\,dx - \int_1^e x\ln(x)\,dx$$

We'll use integration by parts on the second one, with

$u = \ln(x)$ and $v' = x$, so $u' = \dfrac{1}{x}$ and $v = \dfrac{x^2}{2}$.

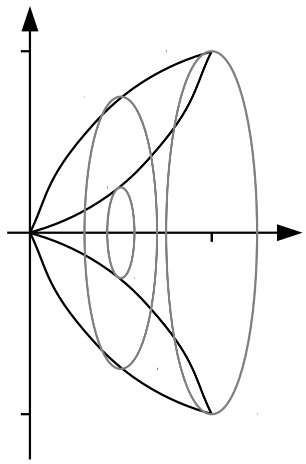
$$\begin{aligned} &= \left.\frac{x^2}{2}\right|_1^e - \int_1^e uv'\,dx = \frac{e^2}{2} - \frac{1^2}{2} - \left[ uv|_1^e - \int_1^e u'v\,dx \right] \\ &= \frac{1}{2}\left(e^2 - 1\right) - \left[ \left.\frac{x^2}{2}\ln(x)\right|_1^e - \int_1^e \frac{1}{x}\cdot\frac{x^2}{2}\,dx \right] \\ &= \frac{1}{2}\left(e^2 - 1\right) - \left[ \frac{e^2}{2} - \frac{1^2}{2} - \frac{1}{2}\int_1^e x\,dx \right] = \frac{1}{2}\left(e^2 - 1\right) - \left[ \frac{1}{2}\left(e^2 - 1\right) - \frac{1}{2}\cdot\left.\frac{x^2}{2}\right|_1^e \right] \\ &= \frac{1}{2}\left(e^2 - 1\right) - \left[ \frac{1}{2}\left(e^2 - 1\right) - \frac{e^2}{4} - \frac{1^2}{4} \right] = \frac{1}{2}\left(e^2 - 1\right) - \left[ \frac{1}{2}\left(e^2 - 1\right) - \frac{1}{4}\left(e^2 - 1\right) \right] \\ &= \frac{1}{2}\left(e^2 - 1\right) - \frac{1}{2}\left(e^2 - 1\right) + \frac{1}{4}\left(e^2 - 1\right) = \frac{1}{4}\left(e^2 - 1\right) \qquad \blacksquare \end{aligned}$$

**Quiz #12.** Friday, 18 January, 2013. *[12 minutes]*

Consider the solid obtained by revolving the region between $y = \sqrt{x}$ and $y = x^2$, for $0 \le x \le 1$, about the $x$-axis.

1. Sketch this solid and find its volume. *[5]*

SOLUTION. Note first that for $0 \le x \le 1$, we have $x^2 \le \sqrt{x}$. Here's a sketch of the solid:



Since we revolved the region about the $x$-axis, the disk/washer method requires that we use $x$ as the basic variable. The (vertical) cross-section of this solid at $x$ is a washer with outer radius $R = \sqrt{x}$ and inner radius (*i.e.* the radius of the whole) $r = x^2$. It follows that the area of the cross-section at $x$ is

$$A(x) = \pi R^2 - \pi r^2 = \pi \left(\sqrt{x}\right)^2 - \pi \left(x^2\right)^2 = \pi x - \pi x^4 = \pi \left(x - x^4\right) ,$$

and so the volume of the solid is

$$\begin{aligned} V &= \int_0^1 A(x)\,dx = \int_0^1 \pi \left(x - x^4\right)\,dx = \pi \int_0^1 \left(x - x^4\right)\,dx \\ &= \pi \left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Bigg|_0^1 = \pi \left(\frac{1^2}{2} - \frac{1^5}{5}\right) - \pi \left(\frac{0^2}{2} - \frac{0^5}{5}\right) \\ &= \pi \left(\frac{1}{2} - \frac{1}{5}\right) - \pi \cdot 0 = \pi \left(\frac{5}{10} - \frac{2}{10}\right) = \frac{3}{10}\pi\,. \end{aligned}$$

Note that the limits of the integral come from the range of $x$'s in the original region. ■

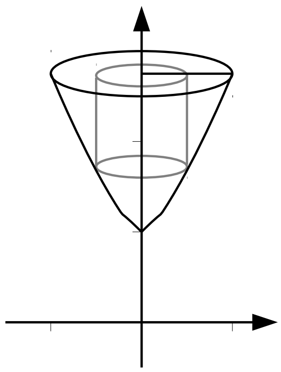
**Quiz #13.** Friday, 25 January, 2013. *[12 minutes]*

Consider the solid obtained by revolving the region between $y = e^x$ and $y = e$, for $0 \le x \le 1$, about the $y$-axis.

1. Sketch this solid and find its volume. *[5]*

SOLUTION. This time we will use the method of cylindrical shells to compute the volume.

Note first that for $0 \le x \le 1$, we have $e^x \le e^1 = e$. Here's a sketch of the solid, with a cylindrical shell drawn in for good measure:



Since we revolved the given region around a vertical line and are using the shell method, we need to use $x$ as the basic variable. The cylindrical shell at $x$ has radius $r = x - 0 = x$ and height $h = e - e^x$. Plugging these into the volume formula for a solid of revolution using shells and integrating away gives us:

$$V = \int_0^1 2\pi r h \, dx = 2\pi \int_0^1 x\left(e - e^x\right) \, dx = 2\pi \int_0^1 ex \, dx - 2\pi \int_0^1 xe^x \, dx$$

We will use integration by parts on the second integral, with $u = x$ and $v' = e^x$, so $u' = 1$ and $v = e^x$.

$$= 2\pi e \frac{x^2}{2}\Big|_0^1 - 2\pi \left[ xe^x|_0^1 - \int_0^1 1e^x \, dx \right]$$

$$= \left( 2\pi e \frac{1^2}{2} - 2\pi e \frac{0^2}{2} \right) - 2\pi \left[ \left(1e^1 - 0e^0\right) - e^x|_0^1 \right]$$

$$= 2\pi e - 2\pi \left[ e - \left(e^1 - e^0\right) \right] = 2\pi e - 2\pi[e - e + 1] = 2\pi(e - 1) \qquad \blacksquare$$

**Quiz #14.** Friday, 8 February, 2013. *[12 minutes]*

1. Compute $\displaystyle\int \frac{2}{x^3 - x}\, dx$. *[5]*

SOLUTION. First, we factor the denominator. $x$ is obviously a factor and pulling it out leaves $x^2 - 1$. Since $1^2 = 1$, this is a difference of squares and so

$$x^3 - x = x\left(x^2 - 1\right) = x(x-1)(x+1)\,.$$

It follows that

$$\frac{2}{x^3 - x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

for some constants $A$, $B$, and $C$. Putting the partial fractions over a common denominator tells us that

$$\begin{aligned}\frac{2}{x^3 - x} = \frac{2}{x(x-1)(x+1)} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \\ &= \frac{A(x-1)(x+1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)} \\ &= \frac{Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx}{x(x-1)(x+1} \\ &= \frac{(A+B+C)x^2 + (B-C)x - A}{x(x-1)(x+1}\,,\end{aligned}$$

so $A + B + C = 0$, $B - C = 0$, and $-A = 2$. Hence $A = -2$, so $B + C = -A = -(-2) = 2$ and $B = C$, so $B = C = 1$. Thus

$$\begin{aligned}\int \frac{2}{x^3 - x}\, dx &= \int \left(\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}\right) dx \\ &= \int \left(\frac{-2}{x} + \frac{1}{x-1} + \frac{1}{x+1}\right) dx \\ &= -2\int \frac{1}{x}\, dx + \int \frac{1}{x-2}\, dx + \int \frac{1}{x+1}\, dx \\ &= -2\ln(x) + \ln(x-1) + \ln(x+1) + K \qquad \blacksquare\end{aligned}$$

**Quiz #15.** Friday, 15 February, 2013. *[25 minutes]*

1. Compute $\displaystyle\int_0^\infty \frac{x^2+3x+3}{(x+1)\left(x^2+2x+2\right)}\,dx$. *[5]*

SOLUTION. First, we work out the antiderivative:

Since $x^2+2x+2=(x+1)^2+1\geq 1$ for all $x$, it is irreducible, so the denominator is given to us fully factored. It follows that

$$\frac{x^2+3x+3}{(x+1)\left(x^2+2x+2\right)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+2x+2}=\frac{A\left(x^2+2x+2\right)+(Bx+C)(x+1)}{(x+1)\left(x^2+2x+2\right)}$$
$$=\frac{(A+B)x^2+(2A+B+C)x+(2A+C)}{(x+1)\left(x^2+2x+2\right)},$$

so $A+B=1$, $2A+B+C=3$, and $2A+C=3$. It follows from the last two equations that $B=3-(2A+B)=3-3=0$, from the first it then follows that $A=1-B=1-0=1$, and then from the last that $C=3-2A=3-2\cdot 1=1$. Thus

$$\int \frac{x^2+3x+3}{(x+1)\left(x^2+2x+2\right)}\,dx=\int\left[\frac{A}{x+1}+\frac{Bx+C}{x^2+2x+2}\right]dx$$
$$=\int\left[\frac{1}{x+1}+\frac{1}{x^2+2x+2}\right]dx$$
$$=\int\frac{1}{x+1}\,dx+\int\frac{1}{x^2+2x+2}\,dx$$
$$=\int\frac{1}{x+1}\,dx+\int\frac{1}{(x+1)^2+1}\,dx$$

We'll use the substitution $u=x+1$, so $du=dx$, in both integrals.

$$=\int\frac{1}{u}\,du+\int\frac{1}{u^2+1}\,du$$
$$=\ln(u)+\arctan(u)+K$$
$$=\ln(x+1)+\arctan(x+1)+K\,.$$

Second, we use this antiderivative to compute the given improper integral:

$$\int_0^\infty \frac{x^2+3x+3}{(x+1)\left(x^2+2x+2\right)}\,dx=\lim_{t\to\infty}\int_0^t \frac{x^2+3x+3}{(x+1)\left(x^2+2x+2\right)}\,dx$$
$$=\lim_{t\to\infty}\left[\ln(x+1)+\arctan(x+1)\right]\Big|_0^t$$
$$=\lim_{t\to\infty}\big(\left[\ln(t+1)+\arctan(t+1)\right]$$
$$-\left[\ln(0+1)+\arctan(0+1)\right]\big)$$
$$=\lim_{t\to\infty}\left(\ln(t+1)+\arctan(t+1)-0-\frac{\pi}{4}\right)$$
$$=\infty\,,$$

since $\ln(t+1)\to\infty$ and $\arctan(t+1)\to\dfrac{\pi}{2}$ as $t\to\infty$. ■

**Quiz #16.** Friday, 1 March, 2013. *[15 minutes]*

Do *one* (1) of questions 1 or 2 below.

1. Find the arc-length of the curve $y = \frac{x^2}{2}$ for $0 \le x \le 1$. *[5]*

2. Find the area of the surface obtained by revolving the curve $y = \frac{x^2}{2}$ for $0 \le x \le 1$ about the $y$-axis. *[5]*

SOLUTION TO 1. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2}{2}\right) = \frac{1}{2} \cdot 2x = x$, so the arc-length is given by:

$$\int_0^1 ds = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1+x^2}\, dx$$

We'll substitute $x = \tan(\theta)$, so $dx = \sec^2(\theta)\, d\theta$ and $\begin{smallmatrix} x & 0 & 1 \\ \theta & 0 & \pi/4 \end{smallmatrix}$.

$$= \int_0^{\pi/4} \sqrt{1+\tan^2(\theta)}\, \sec^2(\theta)\, d\theta = \int_0^{\pi/4} \sqrt{\sec^2(\theta)}\, sec^2(\theta)\, d\theta$$

$$= \int_0^{\pi/4} \sec^3(\theta)\, d\theta = \frac{1}{2}\sec(\theta)\tan(\theta)\Big|_0^{\pi/4} + \frac{1}{2}\ln\left(\sec(\theta)+\tan(\theta)\right)\Big|_0^{\pi/4}$$

$$= \frac{1}{2} \cdot \sqrt{2} \cdot 1 - \frac{1}{2} \cdot 1 \cdot 0 + \frac{1}{2}\ln\left(\sqrt{2}+1\right) - \frac{1}{2}\ln(1+0)$$

$$= \frac{1}{\sqrt{2}} + \ln\left(\sqrt{2}+1\right) \qquad \blacksquare$$

SOLUTION TO 2. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2}{2}\right) = \frac{1}{2} \cdot 2x = x$, and $r = x - 0 = x$ since we are revolving about the $y$-axis, so the surface area is given by:

$$\int_0^1 2\pi r\, ds = \int_0^1 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 2\pi x\sqrt{1+x^2}\, dx$$

We'll substitute $u = 1 + x^2$, so $du = 2x\, dx$ and $\begin{smallmatrix} x & 0 & 1 \\ u & 1 & 2 \end{smallmatrix}$.

$$= \pi \int_1^2 \sqrt{u}\, du = \pi \int_1^2 u^{1/2}\, du = \pi \cdot \frac{2}{3}u^{3/2}\Big|_1^2$$

$$= \frac{2}{3}\pi 2^{3/2} - \frac{2}{3}\pi 1^{3/2} = \frac{2}{3}\pi\left(2\sqrt{2}-1\right) \qquad \blacksquare$$

**Quiz #17.** Friday, 8 March, 2013. *[15 minutes]*

Do *one* (1) of questions 1 or 2 below.

1. Find the arc-length of the polar curve $r = e^\theta$ for $0 \le \theta \le \ln(2)$. *[5]*
2. Find the area of the region between the origin and the polar curve $r = e^\theta$ for $0 \le \theta \le \ln(2)$. *[5]*

SOLUTION TO 1. $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}e^\theta = e^\theta$. Plugging this into the arc-length formula for polar curves gives:

$$\begin{aligned}\int_0^{\ln(2)} ds &= \int_0^{\ln(2)} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{\ln(2)} \sqrt{\left(e^\theta\right)^2 + \left(e^\theta\right)^2}\, d\theta \\ &= \int_0^{\ln(2)} \sqrt{2\left(e^\theta\right)^2}\, d\theta = \sqrt{2}\int_0^{\ln(2)} \sqrt{\left(e^\theta\right)^2}\, d\theta = \sqrt{2}\int_0^{\ln(2)} e^\theta\, d\theta \\ &= \sqrt{2}\cdot e^\theta\Big|_0^{\ln(2)} = \sqrt{2}\left(e^{\ln(2)} - e^\theta\right) = \sqrt{2}\,(2-1) = \sqrt{2} \qquad \blacksquare\end{aligned}$$

SOLUTION TO 2. $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}e^\theta = e^\theta$. Plugging this into the area formula for polar regions gives:

$$\int_0^{\ln(2)} \frac{r^2}{2}\, d\theta = \int_0^{\ln(2)} \frac{\left(e^\theta\right)^2}{2}\, d\theta = \frac{1}{2}\int_0^{\ln(2)} e^\theta e^\theta\, d\theta$$

We'll substitute $w = e^\theta$, so $dw = e^\theta\, d\theta$ and $\begin{matrix}\theta & 0 & \ln(2) \\ w & 1 & 2\end{matrix}$.

$$= \frac{1}{2}\int_1^2 w\, dw = \frac{1}{2}\cdot\frac{w^2}{2}\Big|_1^2 = \frac{1}{4}(4-1) = \frac{3}{4} \qquad \blacksquare$$

**Quiz #18.** Friday, 15 March, 2013. *[10 minutes]*

1. Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n + n}$ converges or diverges. *[5]*

SOLUTION. Since $2^n + n \ge 2^n$ for all $n \ge 0$, we have

$$0 \le \frac{1}{2^n + n} \le \frac{1}{2^n}$$

for all $n \ge 0$. Since $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$ is a geometric series with common ratio $r = \frac{1}{2}$ and $|r| = \frac{1}{2} < 1$, it converges. It follows by the (Basic) Comparison Test that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n + n}$ converges too. $\blacksquare$

**Quiz #19.** Friday, 22 March, 2013. *[10 minutes]*

1. Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n e^n}{n!3^n}$ converges or diverges. *[5]*

Solution. We will use the Ratio Test. Since

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{2^{n+1}e^{n+1}}{(n+1)!3^{n+1}}}{\frac{2^n e^n}{n!3^n}}\right| = \lim_{n\to\infty} \frac{2^{n+1}e^{n+1}}{(n+1)!3^{n+1}} \cdot \frac{n!3^n}{2^n e^n}$$

$$= \lim_{n\to\infty} \frac{2e}{(n+1)3} = \frac{2e}{3} \lim_{n\to\infty} \frac{1}{n+1} = \frac{2e}{3} \cdot 0 = 0 < 1\,,$$

the Ratio Test tells us that the given series converges. (Absolutely, in fact.) ■

**Quiz #20.** Thursday, 28 March, 2013. *[10 minutes]*

1. Determine for which values of $x$ the power series $\displaystyle\sum_{n=0}^{\infty} \frac{nx^n}{5^n}$ converges and for which it diverges. *[5]*

Solution. We will apply the Ratio Test first. Since

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)x^{n+1}}{5^{n+1}}}{\frac{nx^n}{5^n}}\right| = \lim_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{5^{n+1}} \cdot \frac{5^n}{nx^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)x}{5n}\right|$$

$$= \frac{|x|}{5} \lim_{n\to\infty} \frac{n+1}{n} = \frac{|x|}{5} \lim_{n\to\infty} \left(\frac{n}{n} + \frac{1}{n}\right) = \frac{|x|}{5} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)$$

$$= \frac{|x|}{5}(1+0) = \frac{|x|}{5}\,,$$

the series will converge (absolutely) when $\frac{|x|}{5} < 1$, *i.e.* when $|x| < 5$ (that is, $-5 < x < 5$), and diverge when $\frac{|x|}{5} > 1$, *i.e.* when $|x| > 5$ (that is, $x < -5$ or $x > 5$). Unfortunately, the Ratio Test tells us nothing when $\frac{|x|}{5} = 1$, *i.e.* when $x = \pm 5$.

When $x = 5$, the series is $\displaystyle\sum_{n=0}^{\infty} \frac{n5^n}{5^n} = \sum_{n=0}^{\infty} n$. This diverges by the Divergence Test, since $\displaystyle\lim_{n\to\infty} n = \infty \neq 0$.

On the other hand, when $x = -5$, the series is $\displaystyle\sum_{n=0}^{\infty} \frac{n(-5)^n}{5^n} = \sum_{n=0}^{\infty} \frac{n(-1)^n 5^n}{5^n} = \sum_{n=0}^{\infty} (-1)^n n$. This also diverges by the Divergence Test, since $\displaystyle\lim_{n\to\infty} (-1)^n n$ does not exist, and hence $\neq 0$.

It follows that the given series converges exactly when $|x| < 5$, *i.e.* when $-5 < x < 5$, and diverges otherwise. ■

**Quiz #21.** Friday, 5 April, 2013. *[15 minutes]*

1. Find the Taylor series at 0 of $f(x) = \dfrac{x}{1+x} - 1$ and find its radius of convergence. *[5]*

SOLUTION 1. *(Using Taylor's formula.)* We build the usual list of derivatives and values, looking for patterns:

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(0)$ |
|---|---|---|
| 0 | $\frac{x}{1+x} - 1$ | $-1$ |
| 1 | $\frac{1(1+x)-x(1)}{(1+x)^2} - 0 = \frac{1}{(1+x)^2}$ | $1$ |
| 2 | $\frac{-2}{(1+x)^3}$ | $-2$ |
| 3 | $\frac{6}{(1+x)^4}$ | $6$ |
| 4 | $\frac{-24}{(1+x)^5}$ | $-24$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

Looking at the last column, it's not too hard to see that $f^{(n)}(0) = (-1)^{n+1}n!$. Plugging this into Taylor's formula gives the series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}n!}{n!}x^n = \sum_{n=0}^{\infty}(-1)^{n+1}x^n = -1 + x - x^2 + x^3 - \cdots$$

To find the radius of convergence of this Taylor series, we use the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}x^{n+1}}{(-1)^{n+1}x^n}\right| = \lim_{n\to\infty}|-x| = |x|$$

It follows that the series converges (absolutely) for $|x| < 1$ and diverges for $|x| > 1$, so its radius of convergence is $R = 1$. ■

SOLUTION 2. *(Using algebra and cunning ... )* Note that

$$f(x) = \frac{x}{1+x} - 1 = -1 + \frac{x}{1+x} = -1 + \frac{x}{1-(-x)},$$

and the latter part of the last expression is the sum of a geometric series with first term $a = x$ and common ratio $r = -x$. Thus

$$f(x) = -1 + \frac{x}{1-(-x)} = -1 + x - x^2 + x^3 - x^4 + \cdots = \sum_{n=0}^{\infty}(-1)^{n+1}x^n,$$

and since the function is equal to the series (when it converges), this series is the Taylor series of the function at 0.

Since it is a geometric series, it converges exactly when $|r| = |-x| = |x| < 1$, so its radius of convergence is $R = 1$. ■