Mathematics 1101Y – Calculus I: Functions and calculus of one variable
TRENT UNIVERSITY, 2012–2013

Assignment #4
Breaking limits
Due on Friday, 8 February, 2013.

One of the things we've skipped over was the formal definition of limit, that is, how to pin down just what $\lim_{x \to a} f(x) = L$ really means. The usual definition of limits is something like:

> $\varepsilon - \delta$ DEFINITION OF LIMITS. $\lim_{x \to a} f(x) = L$ exactly when for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any x with $|x - a| < \delta$ we are guaranteed to have $|f(x) - L| < \varepsilon$ as well.

Informally, this means that no matter how close – that's the ε – you want $f(x)$ to get to L, you can make it happen by ensuring that x is close enough – that's the δ – to a. If this can always be done, $\lim_{x \to a} f(x) = L$; if not, then $\lim_{x \to a} f(x) \neq L$.

This definition works, but most people find it a little hard to understand and use at first. Here is less common definition equivalent to the one above that is cast in terms of a game:

> LIMIT GAME DEFINITION OF LIMITS. The *limit game* for $f(x)$ at $x = a$ with target L is a three-move game played between two players A and B as follows:
>
> 1. A moves first, picking a small number $\varepsilon > 0$.
> 2. B moves second, picking another small number $\delta > 0$.
> 3. A moves third, picking an x that is within δ of a, *i.e.* $a - \delta < x < a + \delta$.
>
> To determine the winner, we evaluate $f(x)$. If it is within ε of the target L, *i.e.* $L - \varepsilon < f(x) < L + \varepsilon$, then player B wins; if not, then player A wins.
>
> With this idea in hand $\lim_{x \to a} f(x) = L$ means that player B has a winning strategy in the limit game for $f(x)$ at $x = a$ with target L; that is, if B plays it right, B will win no matter what A tries to do. (At least within the rules ... :-) Conversely, $\lim_{x \to a} f(x) \neq L$ means that player A is the one with a winning strategy in the limit game for $f(x)$ at $x = a$ with target L.

Your task in this assignment, should you choose to accept it, is to find such winning strategies:

1. Describe a winning strategy for B in the limit game for $f(x) = 3x - 2$ at $x = 2$ with target 4. Note that no matter what number ε A plays first, B must have a way to find a δ to play that will make it impossible for A to play an x that wins for A on the third move. *[3]*

2. Describe a winning strategy for A in the limit game for $f(x) = 3x - 2$ at $x = 2$ with target 5. Note that A must pick an ε on the first move so that mo matter what δ B tries to play on the second move, A can still find an x to play on move three that wins for A. *[3]*

3. Use either definition of limits above to verify that $\lim_{x \to 1} \left(x^2 + x + 1\right) = 3$. *[4]*

Hint: The choice of δ will probably require some slightly indirect reasoning. Pick some arbitrary positive number, say 1 or 0.5, for δ as a first cut. If it doesn't do the job, but x is at least that close, you'll have some more information to pin down the δ you really need.

NOTE: The problems above are probably easiest done by hand, `Maple` has tools for solving inequalities which could be useful if you should need them.