Trent University

MATH 1101Y Test 2

30 January, 2012

Time: 50 minutes

Name: Solutions

Student Number: 0123456

Question	Mark
1	____
2	____
3	____
4	____
Total	____ /40

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Compute any *three* (3) of the integrals **a**–**f**. *[12 = 3 × 4 each]*

$$\textbf{a.}\ \int_0^1 \left(1 - z^{16}\right)\,dz \quad \textbf{b.}\ \int \frac{e^w + e^{-w}}{2}\,dw \quad \textbf{c.}\ \int_1^e 4x\ln(x)\,dx$$

$$\textbf{d.}\ \int \frac{\sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}\,dx \quad \textbf{e.}\ \int_0^{\pi/4} \cos^2(t)\,dt \quad \textbf{f.}\ \int \frac{1}{\sqrt{9 - x^2}}\,dx$$

SOLUTION TO **a.**

$$\begin{aligned}\int_0^1 \left(1 - z^{16}\right)\,dz &= \int_0^1 1\,dz - \int_0^1 z^{16}\,dz = z\big|_0^1 - \left.\frac{z^{17}}{17}\right|_0^1 \\ &= (1-0) - \left(\frac{1^{17}}{17} - \frac{0^{17}}{17}\right) = 1 - \frac{1}{17} = \frac{16}{17} \quad \blacksquare\end{aligned}$$

SOLUTION TO **b.**

$$\begin{aligned}\int \frac{e^w + e^{-w}}{2}\,dw &= \frac{1}{2}\int e^w\,dw + \frac{1}{2}\int e^{-w}\,dw \\ &\quad \text{Substitute } u = -w, \text{ so } du = (-1)\,dw \text{ and} \\ &\quad dw = (-1)\,du, \text{ in the second integral.} \\ &= \frac{1}{2}e^w + \frac{1}{2}\int e^u(-1)\,du = \frac{1}{2}e^w - \frac{1}{2}e^u + C \\ &= \frac{1}{2}e^w - \frac{1}{2}e^{-w} + C = \frac{e^w - e^{-w}}{2} + C \quad \blacksquare\end{aligned}$$

SOLUTION TO **c.** Use integration by parts, with $u = \ln(x)$ and $v' = 4x$, so $u' = \frac{1}{x}$ and $v = 4\frac{x^2}{2} = 2x^2$.

$$\begin{aligned}\int_1^e 4x\ln(x)\,dx &= 2x^2\ln(x)\big|_1^e - \int_1^e 2x^2\frac{1}{x}\,dx \\ &= \left(2e^2\ln(e)\right) - \left(2\cdot 1^2\ln(1)\right) - \int_1^e 2x\,dx \\ &= 2e^2 - 0 - 2\left.\frac{x^2}{2}\right|_1^e = 2e^2 - \left[e^2 - 1^2\right] = e^2 + 1 \quad \blacksquare\end{aligned}$$

SOLUTION TO **d.** Substitute $u = \sqrt{x} = x^{1/2}$, so $du = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx$.

$$\int \frac{\sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}\,dx = \int \sec^2 u\,du = \tan(u) + C = \tan\left(\sqrt{x}\right) + C \quad \blacksquare$$

SOLUTION TO **e.** We will use the "half-angle" formula $\cos^2(t) = \frac{1}{2} + \frac{1}{2}\cos(2t)$, followed by the substitution $s = 2t$, so $ds = 2\,dt$ and $dt = \frac{1}{2}\,ds$. We will also change the limits when we do the substitution, $\begin{matrix} t & 0 & \pi/4 \\ s & 0 & \pi/2 \end{matrix}$.

$$\begin{aligned}
\int_0^{\pi/4} \cos^2(t)\,dt &= \int_0^{\pi/4} \left(\frac{1}{2} + \frac{1}{2}\cos(2t)\right)\,dt = \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos(s)\right) \cdot \frac{1}{2}\,ds \\
&= \frac{1}{4}\int_0^{\pi/2} (1 + \cos(s))\,ds = \frac{1}{4}\left(s + \sin(s)\right)\Big|_0^{\pi/2} \\
&= \frac{1}{4}\left(\frac{\pi}{2} + \sin\left(\frac{\pi}{2}\right)\right) - \frac{1}{4}\left(0 + \sin(0)\right) = \frac{1}{4}\left(\frac{\pi}{2} + 1\right) - \frac{1}{4}(0+0) \\
&= \frac{\pi}{8} + \frac{1}{4} \qquad \blacksquare
\end{aligned}$$

SOLUTION TO **f.** We will use the trigonometric substitution $x = 3\sin(\theta)$, so $dx = 3\cos(\theta)\,d\theta$ and $\theta = \arcsin\left(\frac{x}{3}\right)$.

$$\begin{aligned}
\int \frac{1}{\sqrt{9 - x^2}}\,dx &= \int \frac{1}{\sqrt{9 - 3^2\sin^2(\theta)}} \cdot 3\cos(\theta)\,d\theta = \int \frac{3\cos(\theta)}{\sqrt{9\left(1 - \sin^2(\theta)\right)}}\,d\theta \\
&= \int \frac{3\cos(\theta)}{\sqrt{9\cos^2(\theta)}}\,d\theta = \int \frac{3\cos(\theta)}{3\cos(\theta)}\,d\theta = \int 1\,d\theta \\
&= \theta + C = \arcsin\left(\frac{x}{3}\right) + C \qquad \blacksquare
\end{aligned}$$

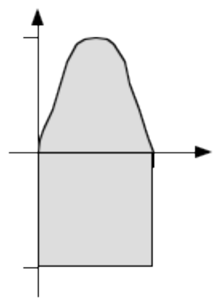
2. Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

a. Sketch the region between $y = \sin(\pi x)$ and $y = -1$, for $0 \le x \le 1$, and find its area.

b. Find the maximum area of a rectangle whose border has total length 36.

c. Use the Right-Hand Rule to compute $\displaystyle\int_0^1 (2x+1)\,dx$.

SOLUTION TO **a.** Here's a very crude sketch of the region:

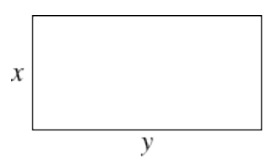


Note that the graph of $y = \sin(\pi x)$ for $0 \le x \le 1$ is just the graph of $\sin(x)$ for $0 \le x \le \pi$ compressed horizontally.

The area of this region is then

$$\begin{aligned}
\int_0^1 (\sin(\pi x) - (-1))\,dx &= \int_0^1 (\sin(\pi x) + 1)\,dx \quad \begin{array}{l}\text{Substitute } u = \pi x, \text{ so} \\ du = \pi\,dx \text{ and } dx = \frac{1}{\pi}\,du, \\ \text{with new limits } \begin{smallmatrix} x & 0 & 1 \\ u & 0 & \pi \end{smallmatrix}.\end{array} \\
&= \int_0^\pi (\sin(u) + 1)\,\frac{1}{\pi}\,du = \left(\frac{1}{\pi}(-\cos(u) + u)\right|_0^\pi \\
&= \frac{1}{\pi}(-\cos(\pi) + \pi) - \frac{1}{\pi}(-\cos(0) + 0) \\
&= \frac{1}{\pi}(-(-1) + \pi) - \frac{1}{\pi}(-1 + 0) = \frac{1}{\pi} + \frac{\pi}{\pi} + \frac{1}{\pi} \\
&= 1 + \frac{2}{\pi}. \quad \blacksquare
\end{aligned}$$

SOLUTION TO **b.** Suppose our rectangle has sides of lengths x and y, respectively.



Then its border has total length $2x + 2y = 36$ and its area is $A = xy$. It follows from the former equation that $y = \dfrac{36 - 2x}{2} = 18 - x$, so $A = xy = x(18 - x) = 18x - x^2$ in terms of x. Note that since any rectangle must have sides of non-negative but finite length, $0 \le x < \infty$. We throw all this information into the procedure for finding maxima and minima:

First, note that $\dfrac{dA}{dx} = \dfrac{d}{dx}\left(18x - x^2\right) = 18 - 2x$. This equals 0 when $x = \dfrac{1}{2} \cdot 18 = 9$, for which value of x we have $A = 18 \cdot 9 - 9^2 = 162 - 81 = 81$.

Second, note that when $x = 0$, $A = 18 \cdot 0 - 0^2 = 0$.

Third, as $n \to \infty$, $A = \left(18x - x^2\right) \to -\infty$ because x^2 grows much faster than x as x increases. (We have to take a limit because ∞, our right "endpoint", isn't a real number.)

Comparing the three values 0, 81, and $-\infty$, the largest, namely 81, gives the maximum area of a rectangle whose total border has length 36. Note that this maximum occurs when $x = y = 9$, *i.e.* when the rectangle is a square with sides of length 9. ■

SOLUTION TO **c.** We plug the given information into the Right-Hand Rule formula,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + i\frac{b-a}{n}\right),$$

and compute away:

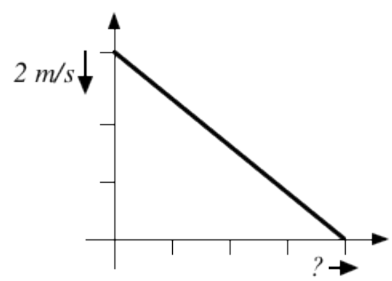
$$\begin{aligned}
\int_0^1 (2x+1)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1-0}{n}\left[2\left(0 + i\frac{1-0}{n}\right) + 1\right] = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\frac{2}{n}i + 1\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[\frac{2}{n}i + 1\right] = \lim_{n\to\infty} \frac{1}{n}\left[\left(\sum_{i=1}^{n} \frac{2}{n}i\right) + \left(\sum_{i=1}^{n} 1\right)\right] \\
&= \lim_{n\to\infty} \frac{1}{n}\left[\frac{2}{n}\left(\sum_{i=1}^{n} i\right) + n\right] = \lim_{n\to\infty} \frac{1}{n}\left[\frac{2}{n} \cdot \frac{n(n+1)}{2} + n\right] \\
&= \lim_{n\to\infty} \frac{1}{n}\left[(n+1) + n\right] = \lim_{n\to\infty} \frac{1}{n}\left[2n + 1\right] = \lim_{n\to\infty}\left[2n\frac{1}{n} + 1\frac{1}{n}\right] \\
&= \lim_{n\to\infty}\left[2 + \frac{1}{n}\right] = 2, \quad \text{since } \frac{1}{n} \to 0 \text{ as } n \to \infty. \blacksquare
\end{aligned}$$

3. Do *one* (1) of **a** or **b**. *[8]*

a. A smooth horizontal floor meets a smooth vertical floor at a right angle, and a ladder 5 m long is set with its base on the floor and its top against the wall and begins to slide down. At the instant that the top of the ladder is 3 m above the floor, the top is moving down at 2 m/s. How is the distance between the base of the ladder and the wall changing at this instant?

b. Sketch the solid obtained by revolving the region below $x + y = 1$ and above $y = 0$ for $0 \leq x \leq 1$ about the y-axis, and find its volume.

SOLUTION TO **a.** We introduce coordinates so that the x-axis lies along the floor and the y-axis along the wall, as in the diagram below.



The ladder, which is 5 m long, forms the hypotenuse of a right-angled triangle in which the other two sides are (parts of) the floor and the wall. If the top of the ladder is at y and the bottom of the ladder is at x, then $x^2 + y^2 = 5^2 = 25$ by the Pythagorean Theorem. Since $y = 3$ at the instant in question, $x = \sqrt{25 - y^2} = \sqrt{25 - 9} = \sqrt{16} = 4$ at this instant. We are given that $\frac{dy}{dt} = -2$ at the same instant.

To obtain $\frac{dx}{dt}$ at the instant in question, we differentiate both sides of $x^2 + y^2 = 25$,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = \frac{d}{dt}\left(x^2 + y^2\right) = \frac{d}{dt}25 = 0\,,$$

plug in what we know of x, y, and $\frac{dy}{dt}$ at the given instant,

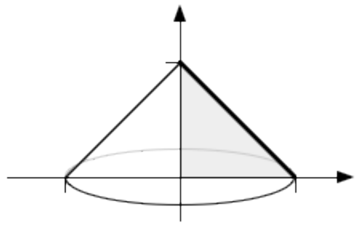
$$8\frac{dx}{dt} - 12 = 2 \cdot 4 \cdot \frac{dx}{dt} + 2 \cdot 3 \cdot (-2) = 0\,,$$

and solve for $\frac{dx}{dt}$:

$$8\frac{dx}{dt} - 12 = 0 \quad \Longrightarrow \quad \frac{dx}{dt} = \frac{12}{8} = \frac{3}{2}$$

Thus the ladder is moving away from the wall (as $\frac{dx}{dt} > 0$) at a rate of $\frac{3}{2}$ m/s at the instant in question. ■

SOLUTION TO **b.** Here is a crude sketch of the solid, with the original region shaded in:



It remains to find the volume of the solid, a cone of height 1 with base radius 1.

Disk/washer method: Since the axis of revolution was the y-axis, we will have to integrate with respect to y; note that the given region includes y values between 0 and 1. The outer radius of the washer at y is $R = x - 0 = x = 1 - y$ (recall that the right border of the region is the line $x + y = 1$). Since the region's left border is the y-axis itself, the inner radius of each washer is $r = 0 - 0 = 0$. We plug all this into the volume formula for the washer method:

$$V = \int_0^1 \pi\left(R^2 - r^2\right)\,dy = \int_0^1 \pi\left((1-y)^2 - 0^2\right)^2\,dy = \pi\int_0^1 \left(1 - 2y + y^2\right)\,dy$$
$$= \pi\left(y - y^2 + \frac{y^3}{3}\right)\Big|_0^1 = \pi\left(1 - 1^2 + \frac{1^3}{3}\right) - \pi\left(0 - 0^2 + \frac{0^3}{3}\right)$$
$$= \pi\left(1 - 1 + \frac{1}{3}\right) - \pi\cdot 0 = \frac{\pi}{3} \qquad \square$$

Cylindrical shell method: Since the axis of revolution was the y-axis, we will have to integrate with respect to x; note that the given region includes x values between 0 and 1. The radius of the cylindrical shell for x is $r = x - 0 = x$, and its height is $h = y - 0 = 1 - x$ (recall that the right border of the region is the line $x + y = 1$). We plug all this into the volume formula for the cylindrical shell method:

$$V = \int_0^1 2\pi r h\,dx = \int_0^1 2\pi x(1-x)\,dx = 2\pi\int_0^1 \left(x - x^2\right)\,dx$$
$$= 2\pi\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = 2\pi\left(\frac{1^2}{2} - \frac{1^3}{3}\right) - 2\pi\left(\frac{0^2}{2} - \frac{0^3}{3}\right)$$
$$= 2\pi\left(\frac{1}{2} - \frac{1}{3}\right) - 2\pi\cdot 0 = 2\pi\frac{1}{6} = \frac{\pi}{3} \qquad \square$$

Volume formula: As noted in class and in the textbook, the volume of a cone of height h and base radius r is $V = \frac{1}{3}\pi r^2 h$. The cone in this problem has $h = 1$ and $r = 1$, so it has a volume of $V = \frac{1}{3}\pi\cdot 1^2\cdot 1 = \frac{\pi}{3}$. ■

4. Find the domain and any (and all!) intercepts, vertical and horizontal asymptotes, local maxima and minima, and points of inflection of $f(x) = xe^{-x}$, and sketch its graph. *[10]*

SOLUTION. Here goes!

i. Domain: e^t is defined and continuous for all real numbers t, so e^{-x} is also defined and continuous for all x. Since x is also defined and continuous everywhere, it follows that $f(x) = xe^{-x}$ is defined and continuous for all x.

ii. Intercepts: $f(0) = 0e^{-0} = 0$, so $f(x)$ has y-intercept 0. Since $e^t > 0$ for all t, $f(x) = xe^{-x} = 0$ can only happen if $x = 0$, so the y-intercept is also the only x-intercept.

iii. Vertical asymptotes: Since $f(x) = xe^{-x}$ is defined and continuous for all x, it cannot have any vertical asymptotes.

iv. Horizontal asymptotes: We check the limits as $x \to \pm\infty$. Recall that as $t \to \infty$, $e^t \to \infty$, and as $t \to -\infty$, $e^t \to 0^+$.

$$\lim_{x\to-\infty} xe^{-x} = \lim_{x\to-\infty} \frac{x}{e^x} \begin{matrix} \to -\infty \\ \to 0^+ \end{matrix} = -\infty \quad \text{since as } x \to -\infty,\ e^x \to 0^+$$

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} \begin{matrix} \to \infty \\ \to \infty \end{matrix} \quad \text{since as } x \to \infty,\ e^x \to \infty$$

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^x} \quad \text{[by l'Hôpital's Rule]} = \lim_{x\to\infty} \frac{1}{e^x} \begin{matrix} \to 1 \\ \to \infty \end{matrix} = 0$$

It follows that $f(x)$ has a horizontal asymptote of $y = 0$ on the right only.

v. Maxima and minima: We first need to compute the derivative.

$$f'(x) = \frac{d}{dx}xe^{-x} = \left(\frac{d}{dx}x\right)e^{-x} + x\left(\frac{d}{dx}e^{-x}\right) = 1e^{-x} + xe^{-x}\frac{d}{dx}(-x)$$
$$= e^{-x} + xe^{-x}(-1) = (1-x)e^{-x}$$

Since $e^x > 0$ for all x, it follows that $f'(x) = (1-x)e^{-x} = 0$ exactly when $1 - x = 0$, *i.e.* when $x = 1$. Moreover, $f'(x) = (1-x)e^{-x} \lessgtr 0$ exactly when $1 - x \lessgtr 0$, *i.e.* when $x \gtrless 1$.

Building the usual table with this information,

x	$(-\infty, 1)$	1	$(1, \infty)$
$f'(x)$	$+$	0	$-$
$f(x)$	$\uparrow$	max	$\downarrow$

,

we see that the maximum at $x = 1$ is the only extreme point of $f(x)$. Note that $f(1) = 1e^{-1} = e^{-1} = \frac{1}{e}$.

vi. Inflection and curvature: We first need to compute the second derivative.

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}(1-x)e^{-x} = \left(\frac{d}{dx}(1-x)\right)e^{-x} + (1-x)\left(\frac{d}{dx}e^{-x}\right)$$
$$= (-1)e^{-x} + (1-x)e^{-x}\frac{d}{dx}(-x) = -e^{-x} + (1-x)e^{-x}(-1)$$
$$= -e^{-x} - e^{-x} + xe^{-x} = (x-2)e^{-x}$$

[Total = 40]

Since $e^x > 0$ for all x, it follows that $f''(x) = (x-2)e^{-x} = 0$ exactly when $x - 2 = 0$, *i.e.* when $x = 2$. Moreover, $f''(x) = (x-2)e^{-x} \lessgtr 0$ exactly when $x - 2 \lessgtr 0$, *i.e.* when $x \lessgtr 2$.

Building the usual table with this information,

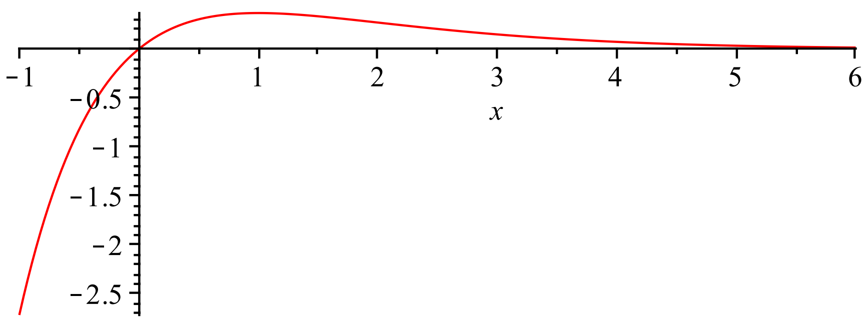
x	$(-\infty, 2)$	2	$(2, \infty)$
$f''(x)$	$-$	0	$+$
$f(x)$	$\frown$	inflection	$\smile$

,

we see that $x = 2$ gives the only inflection point of $f(x)$. Note that $f(2) = 2e^{-2} = \frac{2}{e^2}$.

vii. The graph: Cheating slightly, we plot the graph with `Maple`:

```
> plot(x*exp(-x),x=-1..6);
```



That's all! Whew! ∎