Trent University

MATH 1101Y Test 1

17 October, 2011

Time: 50 minutes

Name: Solutions

Student Number: 00101001

Question	Mark
1	___
2	___
3	___
Total	___ /30

Instructions

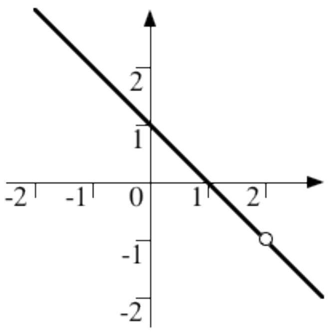
- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

a. Find all the intercepts and the location of the vertex of the parabola $y = x^2 - 2x$ and sketch its graph.

b. Show that $\sin(4x) = 4\sin(x)\cos^3(x) - 4\sin^3(x)\cos(x)$.

c. Find an algebraic expression for the function $f(x)$ whose graph looks like the sketch on the right:
Hint: Note the missing point!



SOLUTION TO **a.** For the y-intercept, we just plug in $x = 0$ and work out the corresponding y value: $y = 0^2 - 2 \cdot 0 = 0$. Thus $(0, 0)$ is the y-intercept.

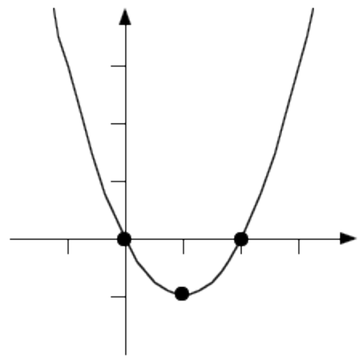
For the x-intercepts, we set $y = 0$ and solve for x: $0 = x^2 - 2x = x(x-2)$, so $x = 0$ or $x = 2$. Thus $(0, 0)$ – which is also the y-intercept! – and $(2, 0)$ are the x-intercepts.

For the location of the vertex, we complete the square in $y = x^2 - 2x$:

$$y = x^2 - 2x = \left(x + \frac{1}{2}(-2)\right)^2 - \left(\frac{1}{2}(-2)\right)^2 = (x-1)^2 - 1$$

It follows that the vertex occurs when $(x-1)^2$ is as small as possible; that is, when $x = 1$. Since $y = -1$ when $x = 1$, the vertex of the parabola is at $(1, -1)$.

Here's a crude sketch:



■

SOLUTION TO **b.** We will make use of the double-angle formulas for both $\sin(\theta)$ and $\cos(\theta)$:

$$\begin{aligned}\sin(4x) &= \sin(2 \cdot 2x) = 2\sin(2x)\cos(2x) = 2 \cdot 2\sin(x)\cos(x)\left(\cos^(x) - \sin^2(x)\right) \\ &= 4\left(\sin(x)\cos^3(x) - \sin^3(x)\cos(x)\right) = 4\sin(x)\cos^3(x) - 4\sin^3(x)\cos(x) \quad ■\end{aligned}$$

SOLUTION TO **c.** Except for the missing point at $x = 2$, the graph is that of the straight line with slope -1 that has y-intercept $(0, 1)$. The equation of this line is $y = -x + 1$. To ensure that it is undefined at $x = 2$ and unchanged everywhere else, we multiply the right-hand side of the equation of the line by $\frac{x-2}{x-2}$, which is undefined at $x = 2$ and equal to 1 everywhere else. Thus the function $f(x)$ whose graph was sketched is $f(x) = (-x+1) \cdot \frac{x-2}{x-2} = \frac{-x^2 + 3x - 2}{x-2}$. ■

2. Do any *two* (2) of **a**–**c**. *[12 = 2 × 6 each]*

a. Determine what kind of discontinuity $f(x) = \dfrac{x^3 - x^2 - 2x}{x+1}$ has at $x = -1$.

b. Compute $\displaystyle\lim_{x \to 0} \frac{\sec^2(x) - 1}{\tan(x)}$.

c. Find all the horizontal asymptote(s) of $f(x) = \dfrac{x^2}{1 + x^2}$.

SOLUTION TO **a.** We need to compute $\displaystyle\lim_{x \to -1^-} f(x)$ and $\displaystyle\lim_{x \to -1^+} f(x)$ and compare them. Since $\dfrac{(-1)^3 - (-1)^2 - 2(-1)}{-1+1} = \dfrac{0}{0}$ – which is why we do have a discontinuity to deal with — we will need to divide $x+1$ into $x^3 - x^2 - 2x$ to be able to compute the limits. (Once we have l'Hôpital's Rule, we'll have a faster way to get the job done ...) One could do this using long division, but in this case it is faster to just factor $x^3 - x^2 - 2x$ because it has an obvious factor, namely x: $x^3 - x^2 - 2x = x\left(x^2 - x - 2\right) = x(x-2)(x+1)$ (Factoring $x^2 - x - 2$ into $(x-2)(x+1)$ is left as an exercise for the reader ... :-)

Using this factorization gives

$$\begin{aligned}
\lim_{x \to -1^-} f(x) &= \lim_{x \to -1^-} \frac{x^3 - x^2 - 2x}{x+1} = \lim_{x \to -1^-} \frac{x(x-2)(x+1)}{x+1} \\
&= \lim_{x \to -1^-} x(x-2) = (-1)(-1-3) = (-1)(-4) = 4 \\
\text{and} \quad \lim_{x \to -1^+} f(x) &= \lim_{x \to -1^+} \frac{x^3 - x^2 - 2x}{x+1} = \lim_{x \to -1^+} \frac{x(x-2)(x+1)}{x+1} \\
&= \lim_{x \to -1^+} x(x-2) = (-1)(-1-3) = (-1)(-4) = 4
\end{aligned}$$

Since both the left- and right-hand limits exist at $x = -1$, and are equal, it follows that $f(x) = \dfrac{x^3 - x^2 - 2x}{x+1}$ has a removable discontinuity at $x = -1$. ■

SOLUTION TO **b.** Rearranging the trig identity $1 + \tan^2(x) = \sec^2(x)$ a little tells us that $\sec^2(x) - 1 = \tan^2(x)$. It follows that

$$\lim_{x \to 0} \frac{\sec^2(x) - 1}{\tan(x)} = \lim_{x \to 0} \frac{\tan^2(x)}{\tan(x)} = \lim_{x \to 0} \tan(x) = \tan(0) = 0\,. \qquad ■$$

SOLUTION TO **c.** We need to compute the limits of $f(x)$ as $x \to -\infty$ and as $x \to +\infty$:

$$\begin{aligned}
\lim_{x \to -\infty} f(x) &= \lim_{x \to -\infty} \frac{x^2}{1+x^2} = \lim_{x \to -\infty} \frac{x^2}{1+x^2} \cdot \frac{1/x^2}{1/x^2} = \lim_{x \to -\infty} \frac{1}{\frac{1}{x^2} + 1} = \frac{1}{0+1} = 1 \\
\lim_{x \to +\infty} f(x) &= \lim_{x \to +\infty} \frac{x^2}{1+x^2} = \lim_{x \to +\infty} \frac{x^2}{1+x^2} \cdot \frac{1/x^2}{1/x^2} = \lim_{x \to +\infty} \frac{1}{\frac{1}{x^2} + 1} = \frac{1}{0+1} = 1
\end{aligned}$$

(Note that $\frac{1}{x^2} \to 0$ as $x \to \pm\infty$.) It follows that $f(x)$ has the line $y = 1$ as a horizontal asymptote in both directions. ■

3. Do *one* (1) of **a** or **b**. *[8]*

a. Suppose $f(x) = \begin{cases} cx+1 & \text{if } x \leq 0 \\ x-c & \text{if } x > 0 \end{cases}$ is continuous at $x = 0$. What must c be?

b. Find the inverse function $f^{-1}(x)$ of $f(x) = \ln(\sin(x)) - \ln(\cos(x))$.

SOLUTION TO **a.** To be continuous at $x = 0$, we need to have $\lim_{x \to 0^-} f(x) = f(0) = \lim_{x \to 0^+} f(x)$. Looking at the definition of $f(x)$, we see that $f(0) = c \cdot 0 + 1 = 0 + 1 = 1$. Computing the limits, we get:

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} (cx + 1) = c \cdot 0 + 1 = 0 + 1 = 1 \\ \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} (x - c) = 0 - c = -c \end{aligned}$$

Thus, in order for $f(x)$ to be continuous, we need to have $1 = 1 = -c$, from which it follows that that $c = -1$. ∎

SOLUTION TO **b.** As usual, since $y = f(x)$ if and only if $x = f^{-1}(y)$, we try to solve for x in terms of y in $y = f(x)$. Here goes:

$$\begin{aligned} & y = \ln(\sin(x)) - \ln(\cos(x)) = \ln\left(\frac{\sin(x)}{\cos(x)}\right) = \ln(\tan(x)) \\ \iff \quad & e^y = e^{\ln(\tan(x))} = \tan(x) \\ \iff \quad & \arctan(e^y) = x \end{aligned}$$

It follows that $x = f^{-1}(y) = \arctan(e^y)$, so $f^{-1}(x) = \arctan(e^x)$. ∎

Too easy? Too hard? Just right?
In any case, it's over!

[Total = 30]